Generating non-periodic tilings in the form of a spiral by using a decorated monotile

Mike Winkler

Fakultät für Mathematik
Ruhr-Universität Bochum, Germany
mike.winkler@ruhr-uni-bochum.de
www.mikematics.de

September 3, 2019

Abstract

It is conjectured that a non-periodic tiling of the plane with a set of four prototiles can be realised by a simple decorated monotile and a simple replication-algorithm in a spiral movement. The algorithm uses overlaps and gaps, but there is no need of adding or removing edges. We also show some mutually locally derivable tilings including a set of two asymmetric prototiles never shown before (see Figure 1).

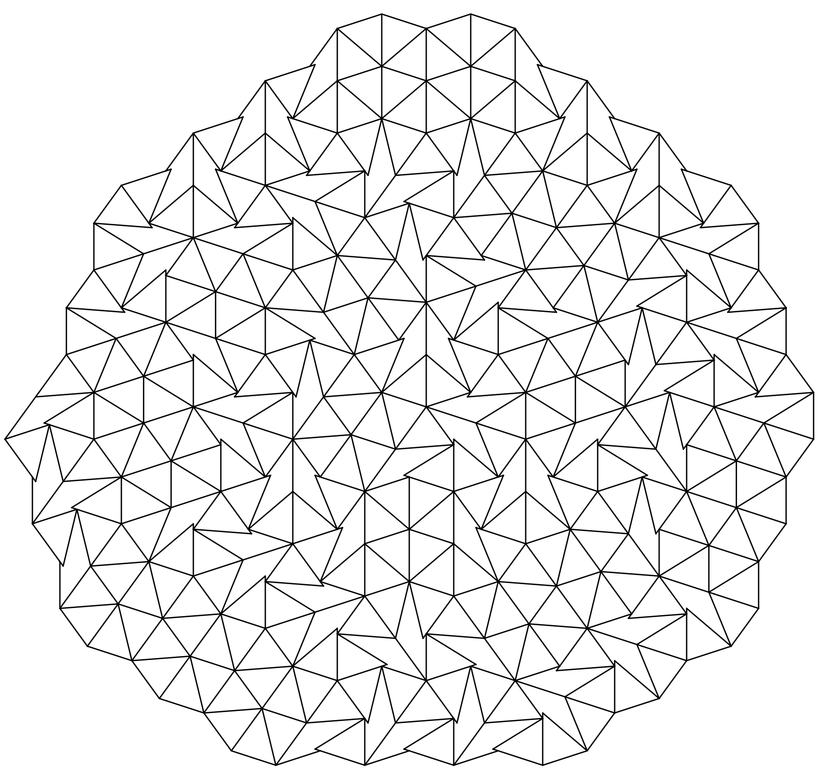


Figure 1: Beginning of a non-periodic spiral tiling with a set of two asymmetric prototiles.

1 Introduction

In geometry, a tiling is a partition of the plane into closed sets (tiles), without overlaps or gaps. A tiling is considered periodic if there exist translations in two independent directions which map the tiling onto itself. Such a tiling is composed of a single fundamental unit or primitive cell which repeats endlessly and regularly in two independent directions. A tiling that cannot be constructed from a single primitive cell is called non-periodic. If a given set of tiles allows only non-periodic tilings, then this set of tiles is called aperiodic. The tilings obtained from an aperiodic set of tiles are often called aperiodic tilings, though strictly speaking it is the tiles themselves that are aperiodic. The tiling itself is said to be ”non-periodic”. [4]

Figure 2 shows the beginning of a non-periodic tiling. The tiling is composed of a set of four prototiles called blossom, starfish, hexagon, and snitch (see Figure 3). Because these prototiles can form also periodic tilings, the set itself is not aperiodic (see Figure 4). In this paper we give a simple algorithm which generates these non-periodic tiling in a spiral movement using a decorated monotile T_m which is an arrangement of a blossom and five starfishes (see Figure 5). The algorithm duplicates T_m in each step (beginning in the center) by mirroring it at one of the ten free outside edges of the blossom. Each figure showing the beginning of a spiral tiling gives the first 68 steps of the algorithm. Note that T_m solves not the ”einstein problem”, because the algorithm uses overlaps and gaps. But there is no need of adding or removing edges.

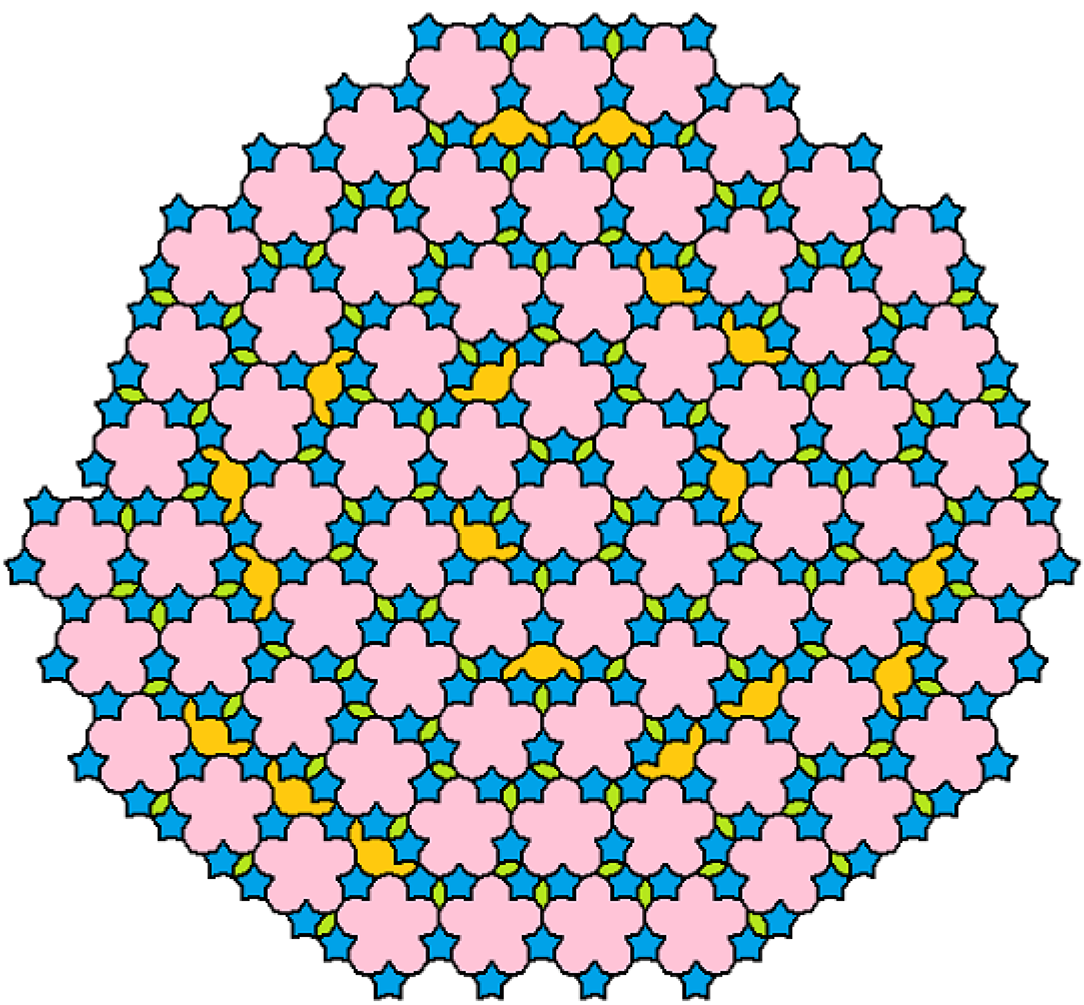


Figure 2: Beginning of a non-periodic tiling with four prototiles. Blossom (pink), starfish (blue), hexagon (green), and snitch (yellow).

2 Blossom, starfish, hexagon, and snitch

Figure 3 shows the four prototiles blossom, starfish, hexagon, and snitch. The edges have all unit length and the angles measure 36° or a multiple of it. The blossom and the starfish have a rotational symmetry of order 5. The hexagon and the snitch are mirror symmetric. The snitch is named referring to the ”golden snitch” of the well-known Harry Potter Stories. These set of prototiles is not aperiodic, because also periodic tilings are possible. For example, the decorated monotile from Figure 4a allows a simple periodic tiling based on a grid of rectangles as seen in Figure 4b.

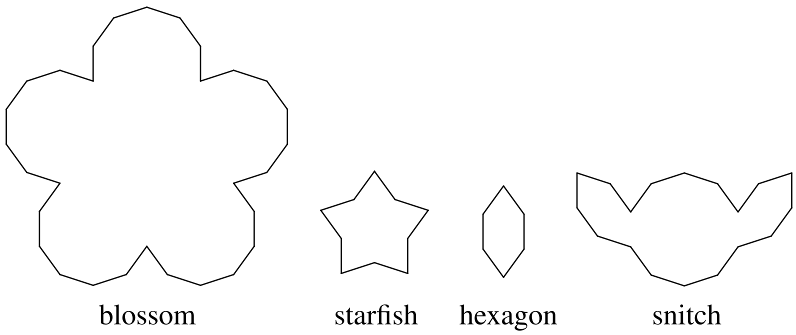
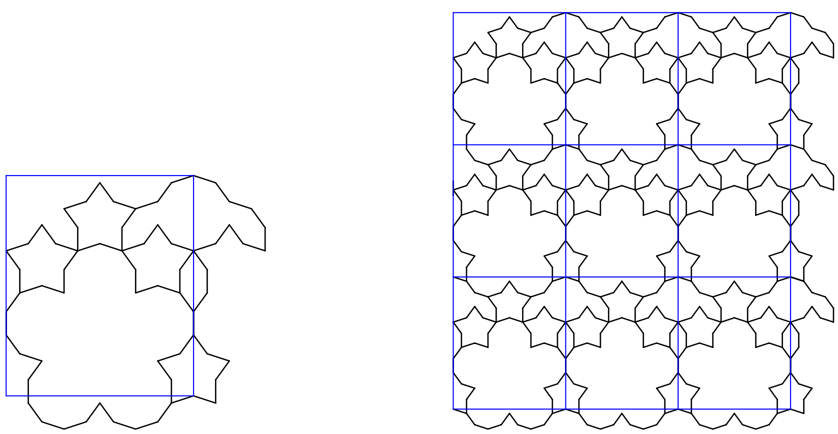


Figure 3: Set of four prototiles with unit length edges.



(a) (b)

Figure 4: Periodic tiling with the tile set from Figure 3.

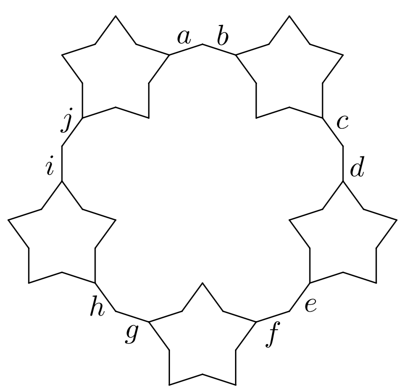


Figure 5: The monotile T_m. A blossom surrounded by five starfishes.

3 The spiral tiling algorithm

Given is the monotile T_m shown in Figure 5. The algorithm mirrors T_m in each step at one of the ten edges $E \in \{a, b, c, d, e, f, g, h, i, j\}$. Let n be the number of steps, and let $T_m(n+1)$ the mirrored copy of $T_m(n)$. The mirroring step $T_m(n)$ at the edge E we write as $T_m(n+1) = E$.

The initial step is given by $T_m(0)$, and the first step is given by $T_m(1) = j$. Now for each $n \geq 1$, $T_m(n)$ will be mirrored in a clockwise spiral movement as close as possible to the already existing tiles $T_m(0), \ldots, T_m(n-1)$ with overlaps and gaps, but without cutting the shape of any blossom or starfish. The generated gaps form only snitch and hexagon tiles. Figure 6 shows the first 68 steps of the algorithm. In Figure 7 we give the first six steps in detail.

For $n \geq 1$ we get the sequence: $T_m(n) = j, b, i, c, j, d, j, d, j, d, j, d, i, c, i, c, h, b, h, c, i, d, j, d,$
$i, c, h, b, g, a, f, j, e, i, h, c, g, b, g, c, h, i, e, j, e, i, d, h, c, \ldots$

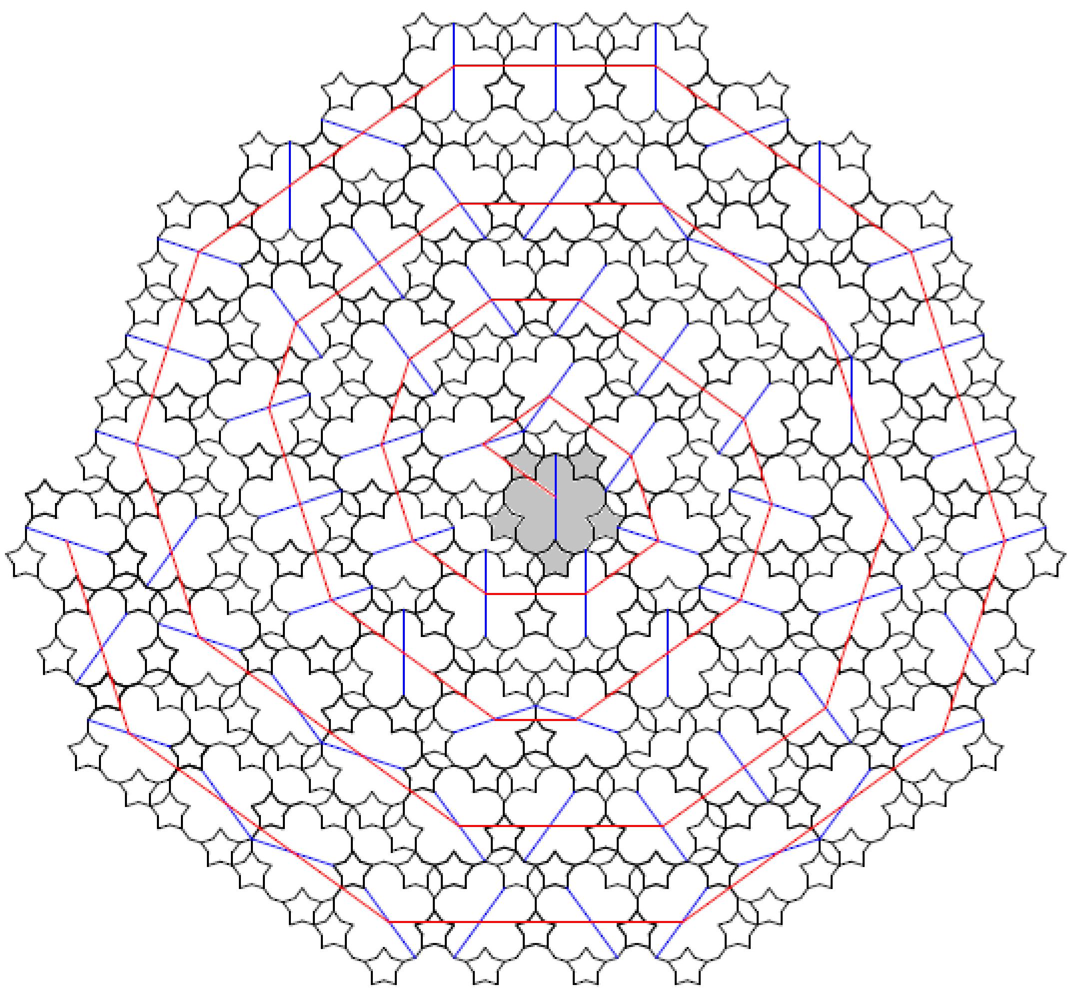


Figure 6: Beginning of a non-periodic tiling with four prototiles generated from 68 mirrored copies of the centered monotile T_m (gray) in a spiral movement (red). The blue lines show the mirroring orientation of the tiles.

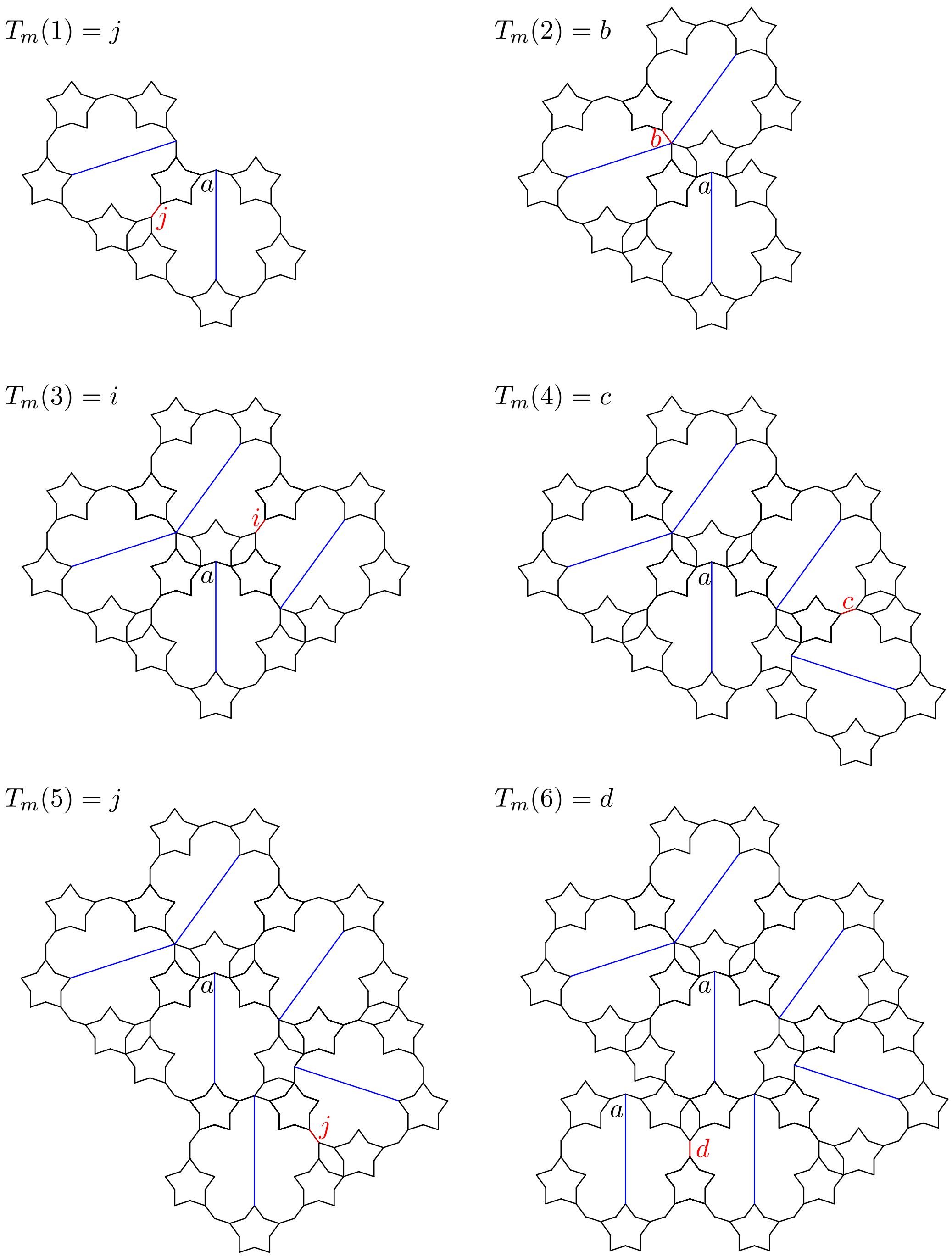


Figure 7: The first six steps of the spiral tiling algorithm.

If $n = 33$ each of the ten mirroring edges $(a, b, c, d, e, f, g, h, i, j)$ has been used at least one time. The initial position of $T_m(0)$, as seen in Figure 5, is reached again if $n = 4k + 6$ for every integer $k \geq 0$. For example, for $k = 0$ we have $T_m(6)$ as seen in Figure 7.

4 Mutually locally derivable tilings

Two tilings are said to be mutually locally derivable (MLD) from each other, if one tiling can be obtained from the other by a simple local rule (such as deleting or inserting an edge) [4]. Our tiling is MLD to Girih tiles (decagon, hexagon, bow), and because of this, also MLD to Robinson triangles. For more information about Girih tiles we refer the reader to [2], [4] and [5].

4.1 Girih tiles

The blossom can be substituted by a starfish and five decagons (Fig. 8a). The starfish can be substituted by an ivy leaf and a bow (Fig. 8b). The snitch can be substituted by a decagon and two hexagons (Fig. 8c). The decagon can be substituted by an ivy leaf and two hexagons (Fig. 8d). The ivy leaf can be substituted by a hexagon and a bow (Fig. 8e). Its not hard to verify that our tiling can be reduced to only hexagon and bow tiles. Substitutions for the hexagon and the bow by Robinson triangles are given in Figure 10.

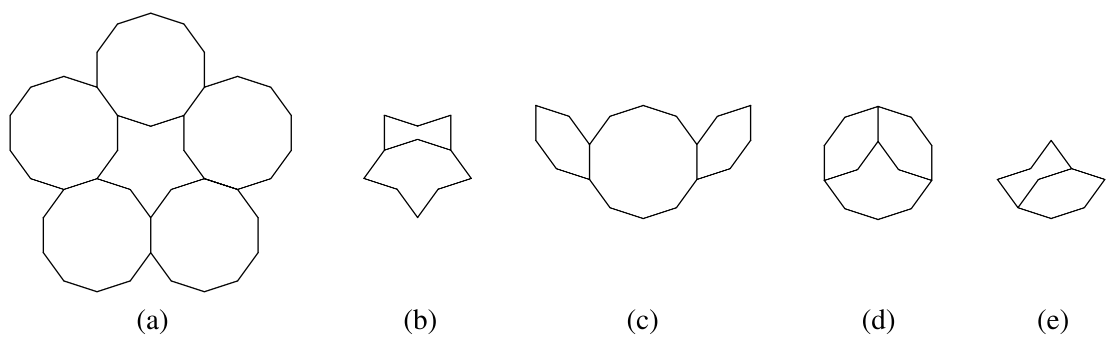


Figure 8: Substitutions with Girih tiles for blossom, starfish, snitch, decagon, and ivy leaf.

4.2 Robinson triangles

A golden triangle is an isosceles triangle in which the duplicated side is in the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ to the distinct side. In Figure 9 we have the golden triangle ABC which can be bisected in the so-called Robinson triangles: the golden gnomon AXC and the golden triangle XBC. The same is true for a golden gnomon. A golden gnomon and a golden triangle with their equal sides matching each other in length, are also referred to as the obtuse and acute Robinson triangles.

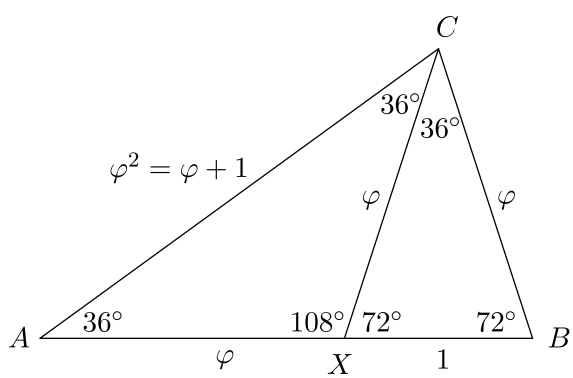
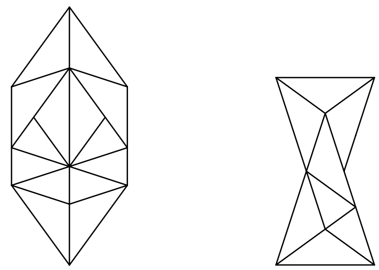


Figure 9: Golden triangle bisected in Robinson triangles: golden gnomon and golden triangle .

The distance of AX and CX are both equal to φ, as seen in the figure. The golden triangle has a ratio of base length to side length equal to the golden section φ, whereas the golden gnomon has the ratio of side length to base length equal to the golden section φ. A Penrose dart is made from AXC mirrored at XC, and a Penrose kite is made from XBC mirrored at XC. [3]

There exist many of different rules and combinations for substituting the blossom, starfish, and snitch tiles by decagons, ivy leafs, hexagons, and bows. We leave it to the interested reader to create his own substitution rules and non-periodic tilings using the spiral tiling algorithm.



(a) (b)

Figure 10: Substitutions with Robinson triangles for hexagon and bow.

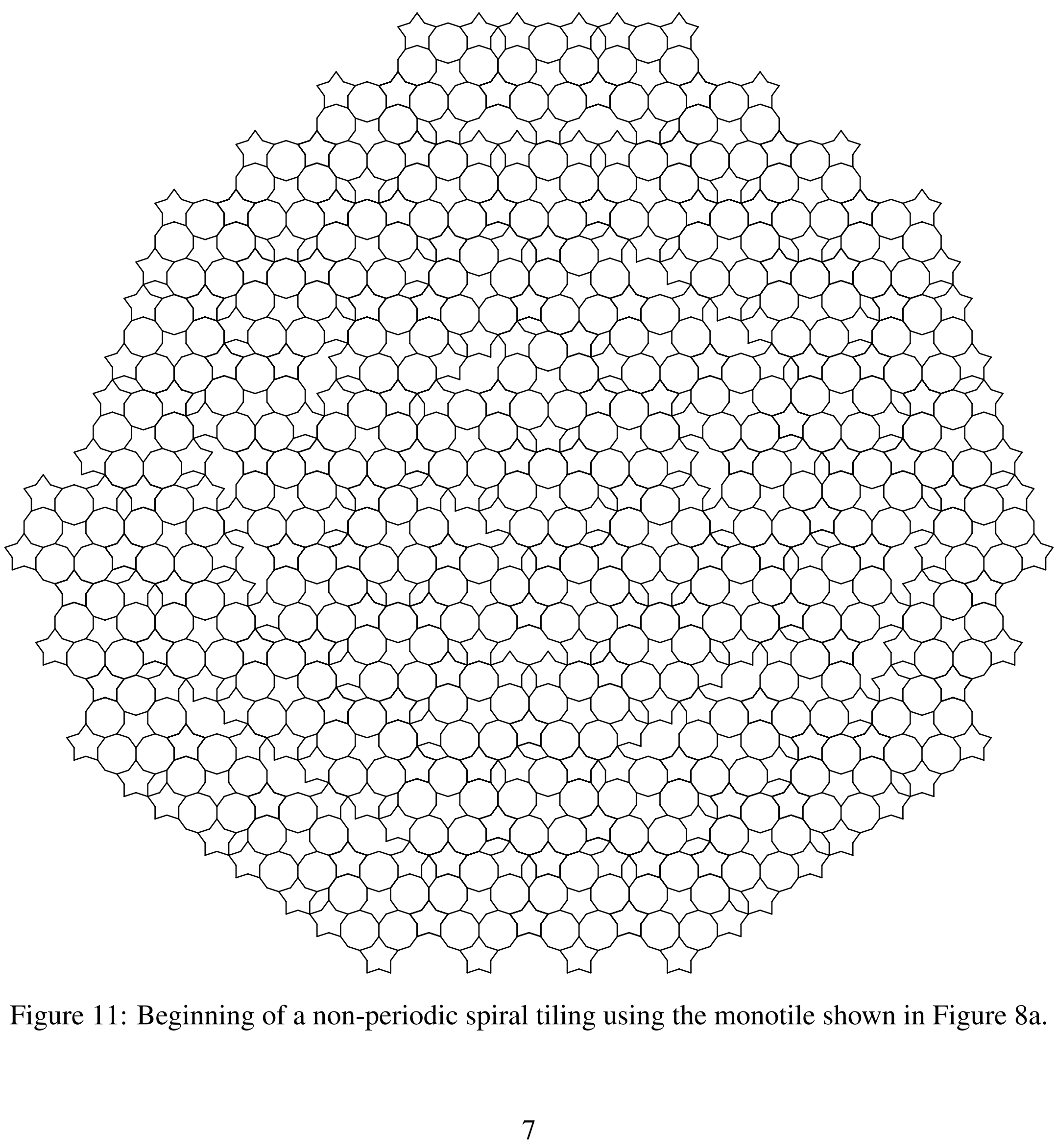


Figure 11: Beginning of a non-periodic spiral tiling using the monotile shown in Figure 8a.

The algorithm also works with a simple decagon monotile with edge lengths of φ. The so generated non-periodic tiling contains only of decagon, hexagon, and bow tiles (see Figure 12), but needs additional edges of length φ in certain areas. These areas are the places where two or more snitch tiles lie next to each other as can be seen in Figure 2 and 6.

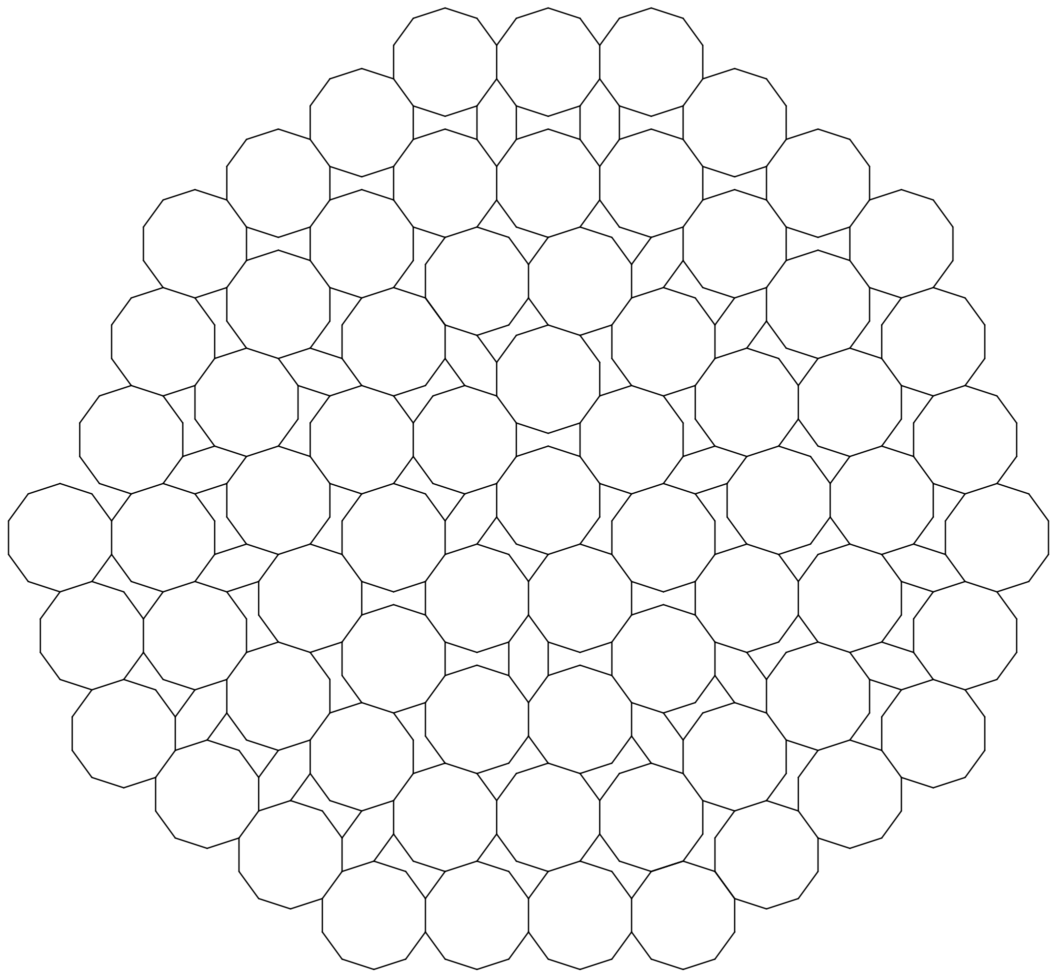


Figure 12: Beginning of a non-periodic spiral tiling using a simple decagon as monotile and additional edges.

If we use another variation of T_m decorated with five additional centered edges of length φ^2 (see Figure 13) we get the tiling from Figure 14.

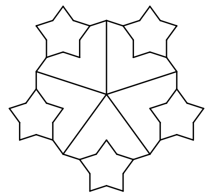


Figure 13: The monotile T_m with additional edges.

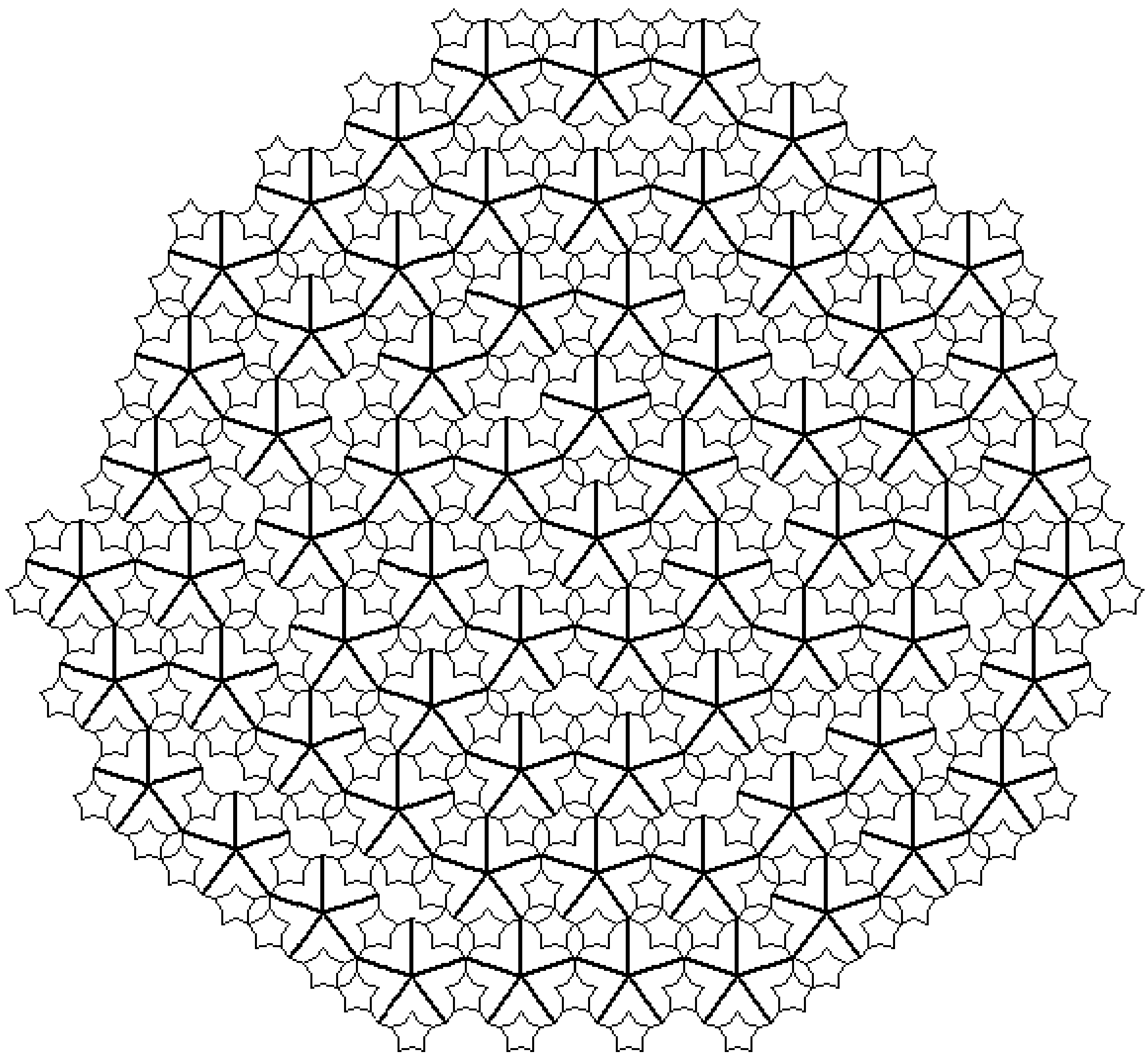
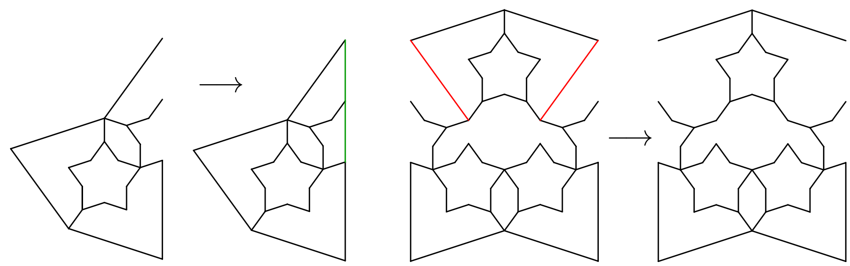


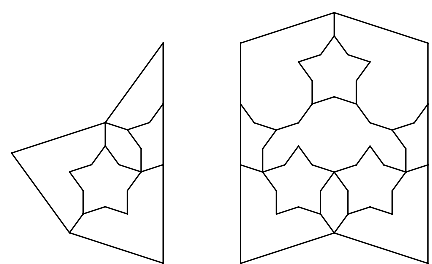
Figure 14: Beginning of a non-periodic spiral tiling generated with the monotile from Figure 13.

From the tiling in Figure 14 we will extract two decorated prototiles (see Figure 16) by using two simple rules of adding and removing edges. We add an edge of length 2φ as seen in Figure 15a, and remove two edges of length φ^2 as seen in Figure 15b. After applying these rules on the tiling from Figure 14 we get the tiling from Figure 18.



(a) (b)

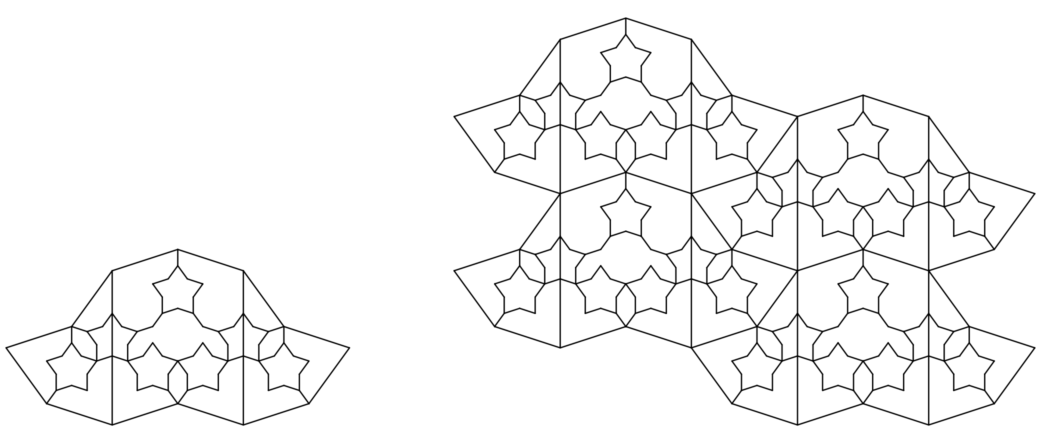
Figure 15: Rules for adding and removing edges.



(a) (b)

Figure 16: Set of two decorated prototiles.

Tile 16a is asymmetric and has the shape of a half ivy leave. Tile 16b is mirror symmetric, has the shape of a broad arrow and can be bisected in four congruent parallelograms with side lengths of φ^2 and $\frac{3\varphi+1}{2}$ (see Figure 19b). These tiles are the largest prototiles (set of two) in terms of area we can use for our non-periodic tiling (see Figure 18). The set itself is not aperiodic, because it allows also simple periodic tilings. For example, with the substitution rule $(a) \longrightarrow (b)$ given in Figure 17.



(a) (b)

Figure 17: Periodic tiling with the tile set from Figure 16.

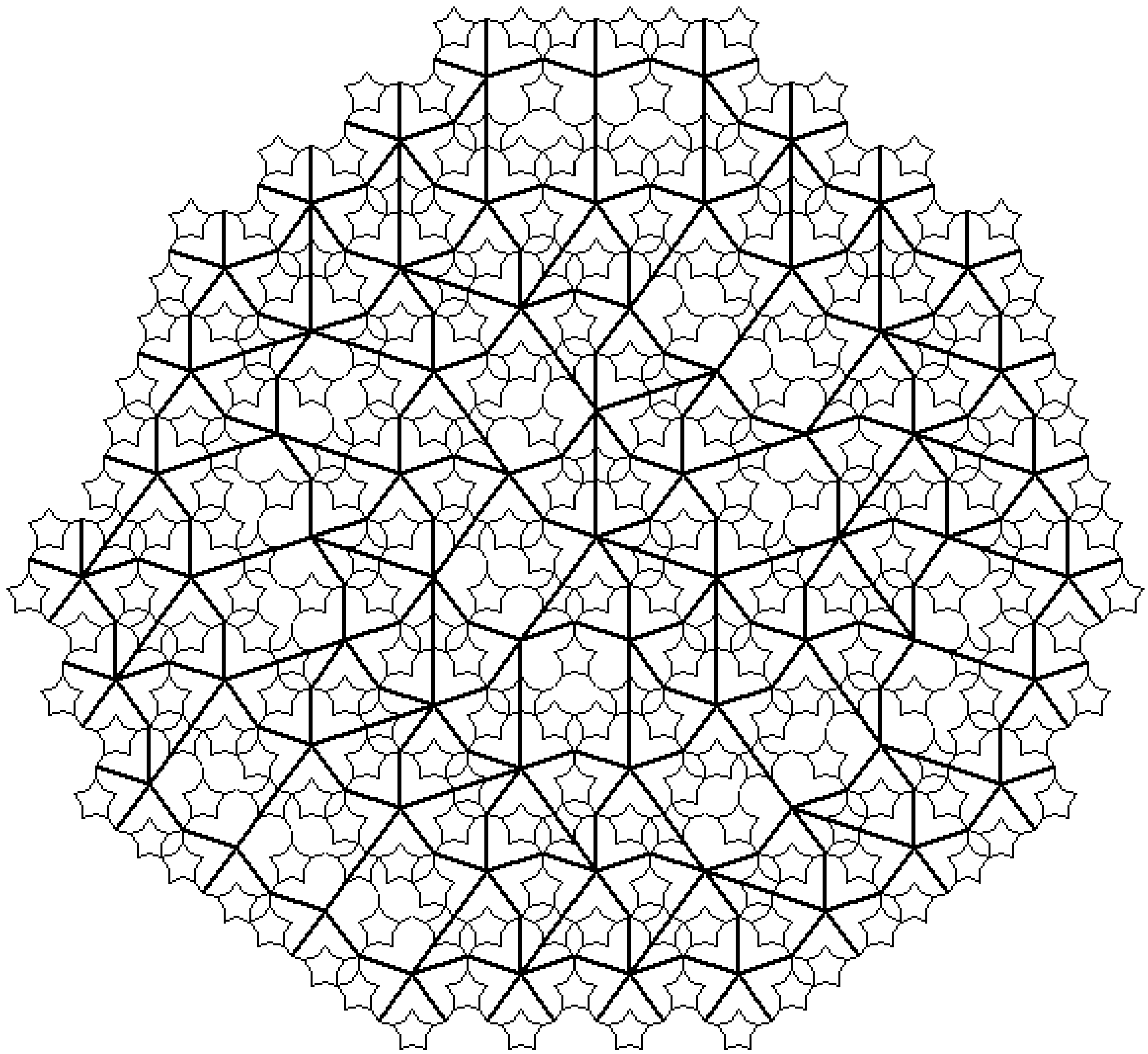
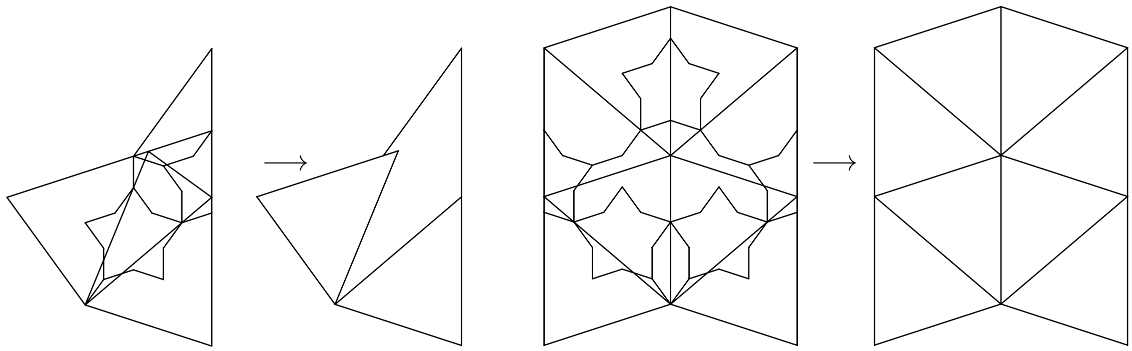


Figure 18: Beginning of a non-periodic spiral tiling with the set of two prototiles from Figure 16.

4.3 Two new prototiles

The half ivy leave can be bisected in three asymmetric tiles, two congruent triangles and a flash shaped tile. Eight of these triangles can build the broad arrow tile from Figure 16b (see Figure 19 and 20).



(a) (b)

Figure 19: Extracting two new prototiles.

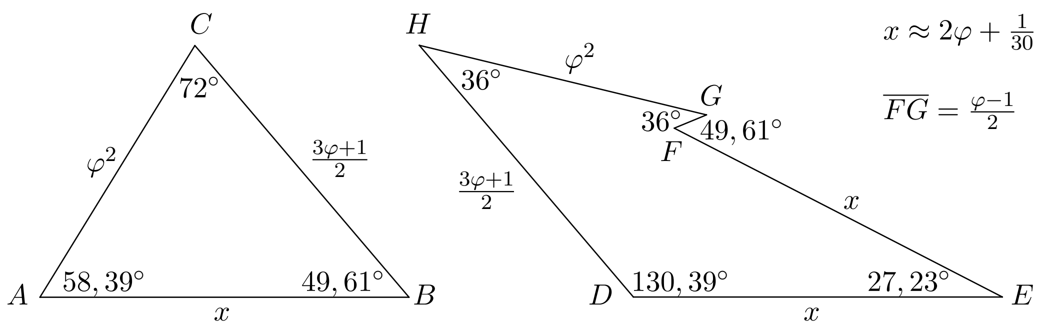


Figure 20: Set of two prototiles.

With the cosine formula we have

$$
\begin{aligned}
x &= \sqrt{\left(\frac{3\varphi+1}{2}\right)^2 + \varphi^4 - 2\varphi^2\left(\frac{3\varphi+1}{2}\right)\cos 72} \\
&= \sqrt{\frac{9}{4}\varphi^4 - (\varphi^4 + 2\varphi^3)\cos 72} \\
&\approx 3,2689 \\
&\approx 2\varphi + \frac{1}{30}.
\end{aligned}
$$

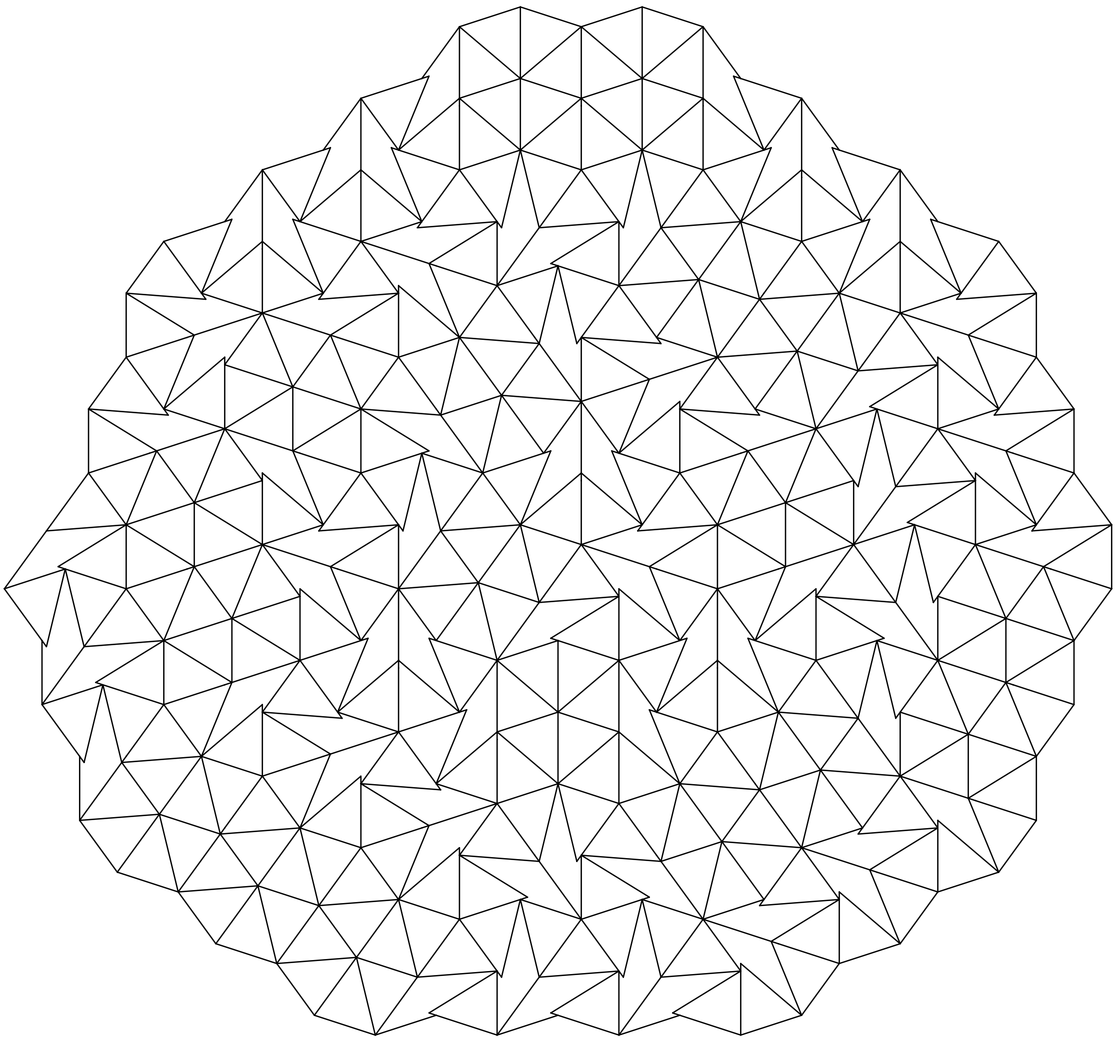


Figure 21: Beginning of a non-periodic spiral titling with the set of two prototiles from Figure 20.

Because the ivy leaf allows simple periodic tilings, the tile set of Figure 20 itself is not aperiodic (see Figure 22).

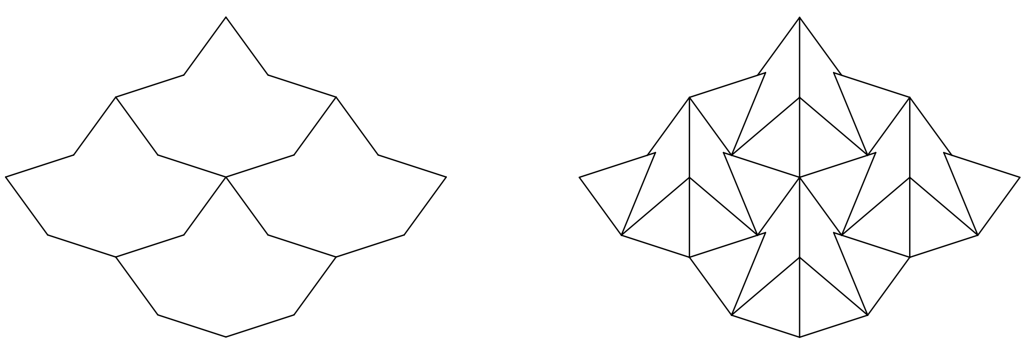


Figure 22: Periodic tiling with the tile set from Figure 20.

At least we give substitution rules with Robinson triangles for the decorated prototiles of Figure 16.

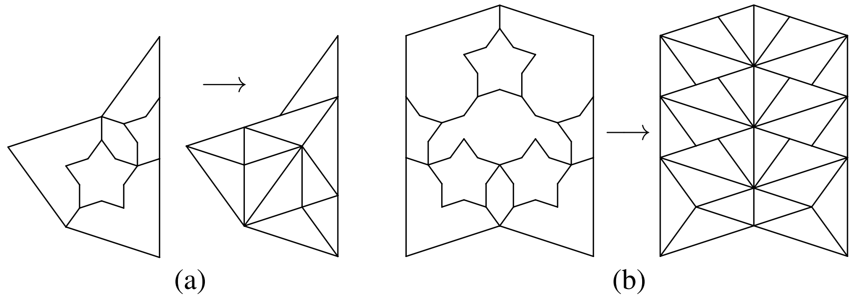


Figure 23: Substitution rules with Robinson triangles.

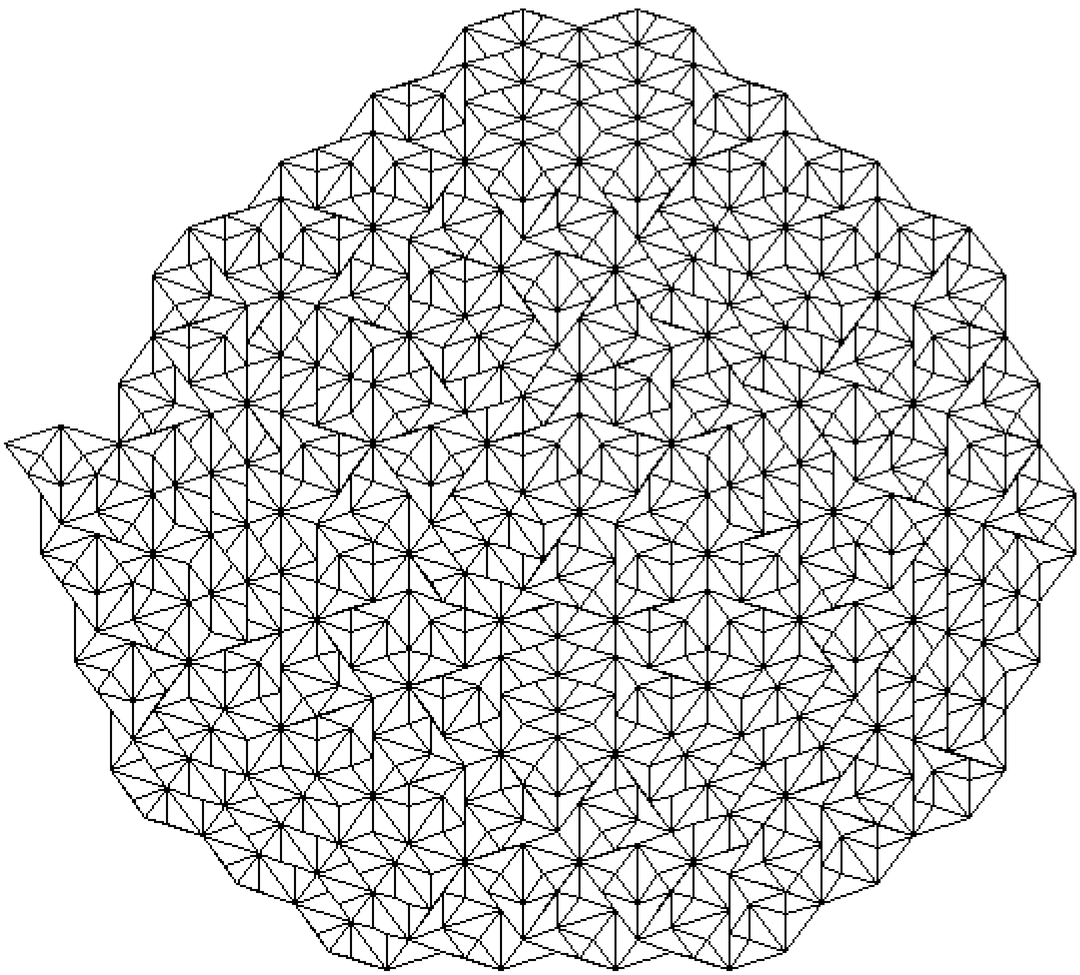


Figure 24: Beginning of a non-periodic spiral tiling with a set of two Robinson trinangles.

It remains an open question if there are substitution rules for the sets of prototiles for generating the presented tilings without the spiral algorithm.

5 References

[1] Wikipedia, *Aperiodic tiling*, September 2019.

[2] Wikipedia, *Girih tiles*, September 2019.

[3] Wikipedia, *Golden triangle (mathematics)*, September 2019.

[4] Wikipedia, *List of aperiodic sets of tiles*, September 2019.

[5] Tilings Encyclopedia, September 2019. https://tilings.math.uni-bielefeld.de/