ON A STOPPING TIME ALGORITHM OF THE 3n + 1 FUNCTION

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ABSTRACT. The behaviour of a Collatz sequence is clearly related to the way in which the powers of 2 are distributed among the powers of 3 in a stopping time term formula. It is shown, that the seemingly chaotically distribution of the powers of 2 can be generated by an iterative algorithm (conjecture), which also allows an insight into the properties of the number $\log_2 3$. From Chapter 3 we are interested in only the odd starting numbers of the Collatz function. Chapter 1. – 3. of the this text are based on a text by Lynn E. Garner [1].

1. Introduction. The Collatz 3n + 1 function is defined as a function T: $\mathbb{N} \rightarrow \mathbb{N}$ on the set of positive integers by

$$T(n) := \begin{cases} T_0(n) := \dfrac{n}{2} & \text{if } n \text{ is even,} \\ T_1(n) := \dfrac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

Let $T^0(s) = s$ and $T^k(s) = T(T^{k-1}(s))$ for $k \in \mathbb{N}$. The Collatz sequence for $s \in \mathbb{N}$ is

$$C(s) = \{T^k(s) \mid k = 0, 1, 2, \ldots\}.$$

For example, C(11) = {11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, 2, 1, 2, 1, … }.

2. Stopping time. Collatz's conjecture is equivalent to the conjecture that for each $s \in \mathbb{N}$, $s > 1$, there exists $k \in \mathbb{N}$ such that $T^k(s) < s$. The least $k \in \mathbb{N}$ such that $T^k(s) < s$ is called the stopping time of s, which we will denote by $\sigma(s)$. It is not hard to verify that

$\sigma(s) = 1$ if $s \equiv 0 \pmod 2$,
$\sigma(s) = 2$ if $s \equiv 1 \pmod 4$,
$\sigma(s) = 4$ if $s \equiv 3 \pmod{16}$,
$\sigma(s) = 5$ if $s \equiv 11, 23 \pmod{32}$,
$\sigma(s) = 7$ if $s \equiv 7, 15, 59 \pmod{128}$,
$\sigma(s) = 8$ if $s \equiv 39, 79, 95, 123, 175, 199, 219 \pmod{256}$,

and so forth.

T. D. Noe [2] has listed these numbers. J. C. Everett [3] proves that almost all $s \in \mathbb{N}$ have finite stopping time, and R. Terras [4] gives a probability distribution function for stopping times.

Let $\kappa(u) = \lceil u \cdot \log_2 3 \rceil$, so it is proofed, that for each $u \in \mathbb{N}$ there exists a set of r odd positive integers n_z, z = 1, …, r, defined by

$$\left.\begin{aligned} &N_u := \{n_z \in \mathbb{N} \mid n_1 < \ldots < n_r < 2^{\kappa(u)}\}, \\ \text{so that}\quad &\sigma(s) = \kappa(u) \ \text{ if } \ s \equiv n_z \ (\text{mod } 2^{\kappa(u)}). \end{aligned}\right\} \tag{2}$$

For simplifying, if z is not needed, we write n_z as n. Let $N^* = \{N_u \mid u \in \mathbb{N}\}$, then all elements of N^* are different.

3. A stopping time term formula for odd numbers. Let $C^k(s)$ consist of the first k terms of the Collatz sequence for odd s, and let u be the number of odd terms in $C^k(s)$. Further let α_i = k, if and only if $T^k(s)$ is odd in $C^k(s)$, then for all $s \equiv n_z \pmod{2^{\kappa(u)}}$ the next term in the Collatz sequence is

$$T^k(s) = \frac{3^u}{2^k} \cdot s + \sum_{i=1}^{u} \frac{3^{u-i} 2^{\alpha_i}}{2^k} < s, \quad k = \kappa(u). \tag{3}$$

In fact of that, if $s \equiv n_z \pmod{2^{\kappa(u)}}$ the (u + 1)th odd term in C(s) is smaller than s.

For example, $C^5(11) = \{11, 17, 26, 13, 20\}$. $T^0 = 11$ is odd, so $\alpha_1 = 0$, $T^1 = 17$ is odd, so $\alpha_2 = 1$, $T^2 = 26$ is even, $T^3 = 13$ is odd, so $\alpha_3 = 3$, $T^4 = 20$ is even. With (3) we get

$$T^5(11) = \frac{3^3}{2^5} \cdot 11 + \frac{3^2 2^0 + 3^1 2^1 + 3^0 2^3}{2^5} = 10 < 11 \cdot$$

4. Binary matrices $\mathbf{B_n}$. For each $u \in \mathbb{N}$ the first u + 1 terms in C(s) represents sufficiently the stopping time of an odd number s, because all the other terms are even before an odd term $T^k(s) < s$ is reached. To simplify the distribution of the powers of 2 in $3^{u-i}2^{\alpha_i}$, let "0" represents an even term and "1" represents an odd term in C(s). For each $u \in \mathbb{N}$ let $B_n[u]$ be a binary (1, u + 1)-matrix, defined by

$$B_n[u] = \begin{pmatrix} b_{1,1} & \cdots & b_{1,u+1} \end{pmatrix} := \begin{pmatrix} b_{1,k+1} := 0 & \text{if } T^k(s) \text{ is even} \\ b_{1,k+1} := 1 & \text{if } T^k(s) \text{ is odd} \end{pmatrix}_{k=0}^{u}. \tag{4}$$

Note that $b_{1,1} = b_{1,2} = 1$ for each $B_n[u]$.

For example, for u = 3 there are two numbers n_z, $n_1 = 11$and $n_2 = 23$. Then
C(11) = {11, 17, 26, 13, 20, 10, 5, …} is represented by $B_{11}[3]$ = (1 1 0 1), and
C(23) = {23, 35, 53, 80, 40, 20, 10, 5, …} is represented by $B_{23}[3]$ = (1 1 1 0).

5. Binary matrices $\mathbf{A_m}$. For $m \in \mathbb{N}$, with $3 \le m \le u - 1$, let A_m be an inverted anti-diagonal binary matrix of m rows and columns, defined by

$$A_m := \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \iddots & \vdots \\ a_{m,1} & \cdots & a_{m,m} \end{pmatrix}, \tag{5}$$

which all entries are "1", except the diagonal from $a_{1,m}$ to $a_{m,1}$, which entries are "0".

For example, $$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

6. Sorting the matrices B_n. For each $u \in \mathbb{N}$ the numbers n_z seem to be distributed chaotically between 0 and $2^{\kappa(u)}$. But if we sort the matrices B_n in a special way, then a system is seen in the arrangement of their elements, which can be transferred into an iterative algorithm.

To understand how the sorting-system is working, let us first take a look at the example for u = 9 on page 12.

For u = 9 there are 173 numbers n_z. With (4) we get 173 matrices B_n, which we can combine into 37 larger matrices S_m such a way, that we get a square-matrix A_m into each matrix S_m. The elements behind A_m of each S_m must be equal for each column.

The index "m" is the number of B_n for each S_m, and also be the number of rows and columns of A_m. The green square frame show A_m.

At least we sort the 37 matrices S_m into 9 sets G_t such a way, that A_m extends a column and a row from bottom to top. The last elements behind A_m must be equal in each set. The biggest matrix S_m of a set G_t contains "0", "0 0" or no more elements behind A_m.

The index "t" of a set G_t is equal to the index "m" of the biggest matrix S_m. In fact of that a set G_t contains t − 2 matrices S_m.

The following chapters 7 and 8 give the mathematical correct definition of the matrices S_m and sets G_t.

7. Binary matrices S_m. For each u > 3 let $S_m[u]$ be a (m, u + 1)-matrix, which rows consist of m *different* matrices $B_n[u]$, defined by

$$S_m[u] := \begin{pmatrix} s_{1,1} & \cdots & s_{1,u+1} \\ \vdots & \ddots & \vdots \\ s_{m,1} & \cdots & s_{m,u+1} \end{pmatrix} = \begin{pmatrix} B_n[u] \\ \vdots \\ B_n[u] \end{pmatrix},$$

which $B_n[u]$ sorted such a way, that

$$\forall \; S_m[u], \begin{pmatrix} s_{1,3} & \cdots & s_{1,m+2} \\ \vdots & \ddots & \vdots \\ s_{m,3} & \cdots & s_{m,m+2} \end{pmatrix} = A_m$$

and $$s_{1,d}, \ldots, s_{m,d} = s_{1,d}, \quad d = m+3, \ldots, u+1. \qquad (6)$$

For example, for u = 9 we have among others $B_{1095}[9] = (1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1)$, $B_{9679}[9] = (1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1)$ and $B_{30715}[9] = (1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1)$. These three matrices can be sorted such a way, that we get by (6):

$$S_3[9] = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} B_{9679}[9] \\ B_{1095}[9] \\ B_{30715}[9] \end{pmatrix}.$$

Note A_3 in columns 3, 4, 5 and the same entries in each column followed.

8. Sets G_t of binary matrices S_m. For each $u > 5$ let $G_t[u]$ be a set of $t - 2$ matrices $S_m[u]$, defined by

$$\left.\begin{aligned}
&G_t[u] := \{S_m[u] \mid m = 3, \dots, t,\ 5 \le t \le u - 1\},\\
&\text{which } S_m[u] \text{ sorted such a way, that for } G_t[u],\ t = 5, \dots, u - 3, \text{ is}\\
&(s_{1,m+5} \cdots s_{1,u+1}) \text{ of } S_m[u] = (s_{1,m+5} \cdots s_{1,u+1}) \text{ of } S_{m+1}[u], \quad m = 3, \dots, t - 1\\
&\text{and for } G_{u-2}[u] \text{ and } G_{u-1}[u] \text{ is}\\
&(s_{1,m+5} \cdots s_{1,u+1}) \text{ of } S_m[u] = (s_{1,m+5} \cdots s_{1,u+1}) \text{ of } S_{m+1}[u], \quad m = 3, \dots, u - 4.
\end{aligned}\right\} \quad (7)$$

See Appendix A for a complete list of sorted matrices $B_n[u]$ for $u = 4, \dots, 9$.

9. Sorting theorem: The numbers of B_n, S_m and G_t for each u. Let $\gamma_k[u]$, $k = 1, \dots, u - 5$, be the number of $G_{u-k}[u]$, then for each $u > 5$ there exists $\gamma_{u-4}[u] \in \mathbb{N}$, defined by

$$\left.\begin{aligned}
&\gamma_{u-4}[u] := \sum_{k=1}^{u-5} \gamma_k[u],\\
&\gamma_1[u] = 1 < \gamma_2[u] < \gamma_3[u] < \dots < \gamma_{u-7}[u] < \gamma_{u-6}[u] = \gamma_{u-5}[u].
\end{aligned}\right\} \quad (8)$$

Let $\gamma_{u-3}[u]$ be the number of all matrices $S_m[u]$ for each $u > 5$, defined by

$$\gamma_{u-3}[u] := \sum_{k=1}^{u-5} \gamma_k[u] \cdot (u - k - 2) \cdot \quad (9)$$

Let $\gamma_{u-2}[u]$ be the number of all matrices $B_n[u]$ for each $u > 5$, defined by

$$\gamma_{u-2}[u] := \sum_{k=1}^{u-5} \frac{\gamma_k[u] \cdot (u - k - 2)(u - k + 3)}{2}. \quad (10)$$

The following Table shows $\gamma_k[u]$ for $u = 6, \dots, 15$. The shaded elements show $\gamma_{u-4}[u]$, $\gamma_{u-3}[u]$, $\gamma_{u-2}[u]$.

u	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}	γ_{13}
6	1	1	3	12									
7	1	1	2	7	30								
8	1	2	2	5	19	85							
9	1	2	3	3	9	37	173						
10	1	3	5	7	7	23	99	476					
11	1	3	6	9	12	12	43	194	961				
12	1	4	9	16	23	30	30	113	525	2652			
13	1	5	14	28	47	66	85	85	331	1570	8045		
14	1	5	15	34	62	99	136	173	173	698	3387	17637	
15	1	6	20	50	103	179	278	377	476	476	1966	9690	51033

For example, $\gamma_1[6] = 1$, $\gamma_2[6] = 1$, $\gamma_3[6] = 3$, $\gamma_4[6] = 12$.

Compare the entries of $u = 6, \dots, 9$ with the sorted matrices on page 10, 11 and 12. Appendix C shows a larger Table of $\gamma_k[u]$ for $u = 6, \dots, 30$.

10. Analyzing the entries of the Table. When analyzing $\gamma_k[u]$, $k \geq 6$, the following relationships are identified.

For u = 11, …, 17

$$\gamma_{u-k}[u] = \gamma_{u-7}[u-5], \quad k \in \{5\}$$

$$\gamma_{u-k}[u] - \gamma_{u-k-1}[u] = \gamma_{u-8}[u-5], \quad k \in \{6, 7, 8\}$$

$$\gamma_{u-k}[u] - 2\gamma_{u-k-1}[u] + \gamma_{u-k-2}[u] = \gamma_{u-9}[u-5], \quad k \in \{8, 9, 10\}$$

$$\gamma_{u-k}[u] - 3\gamma_{u-k-1}[u] + 3\gamma_{u-k-2}[u] - \gamma_{u-k-3}[u] = \gamma_{u-k}[u-5], \quad k \in \{10, 11, 12, 13\}$$

For u = 18, …, 29

$$\gamma_{u-k}[u] = \gamma_{u-7}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{10}[12], \quad k \in \{5\}$$

$$\gamma_{u-k}[u] - \gamma_{u-k-1}[u] = \gamma_{u-8}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{9}[12], \quad k \in \{6, 7, 8\}$$

$$\gamma_{u-k}[u] - 2\gamma_{u-k-1}[u] + \gamma_{u-k-2}[u] = \gamma_{u-9}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{8}[12], \quad k \in \{8, 9, 10\}$$

$$\gamma_{u-k}[u] - 3\gamma_{u-k-1}[u] + 3\gamma_{u-k-2}[u] - \gamma_{u-k-3}[u] = \gamma_{u-k}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{17-k}[12], \quad k \in \{10, \dots, 25\}$$

and so forth. See Appendix D for a larger overview of these relationships.

11. The $\gamma_k[u]$-Algorithm. We conjecture, that exists an iterative algorithm for each $u \geq 11$, which generates $\gamma_1[u], \dots, \gamma_{u-2}[u]$ by using $\gamma_{u-k}[u-w]$ for $k = 7, \dots, u-4$ and $w \geq 5$. Let K_1, K_2, K_3, K_4 be sets of positive integers defined by

$$K_1 := \{5\}, \quad K_2 := \{6, 7, 8\}, \quad K_3 := \{8, 9, 10\}, \quad K_4 := \{10, \dots, u-4\}.$$

Let $\gamma_{u-k}[u] = 0$ for $k \geq u$ and $m = \left\lceil \frac{u-6}{12} \right\rceil$. With $\gamma_k[6], \dots, \gamma_k[10]$ as starting values, the algorithm runs autonomously by using (8), (9), (10) and the following equations. Note that $\gamma_{u-5}[u] = \gamma_{u-6}[u]$.

$$\sum_{i=0}^{0} \binom{1}{i} (-1)^i \gamma_{u-k-i}[u] = \gamma_{u-7}[u-5] - \sum_{n=1}^{m} \gamma_{u-12n-5}[u-12n] \cdot \gamma_{12n-2}[12n], \quad k \in K_1 \tag{11}$$

$$\sum_{i=0}^{1} \binom{1}{i} (-1)^i \gamma_{u-k-i}[u] = \gamma_{u-8}[u-5] - \sum_{n=1}^{m} \gamma_{u-12n-5}[u-12n] \cdot \gamma_{12n-3}[12n], \quad k \in K_2 \tag{12}$$

$$\sum_{i=0}^{2} \binom{2}{i} (-1)^i \gamma_{u-k-i}[u] = \gamma_{u-9}[u-5] - \sum_{n=1}^{m} \gamma_{u-12n-5}[u-12n] \cdot \gamma_{12n-4}[12n], \quad k \in K_3 \tag{13}$$

$$\sum_{i=0}^{3} \binom{3}{i} (-1)^i \gamma_{u-k-i}[u] = \gamma_{u-k}[u-5] - \sum_{n=1}^{m} \gamma_{u-12n-5}[u-12n] \cdot \gamma_{12n+5-k}[12n], \quad k \in K_4 \tag{14}$$

Appendix B shows a Delphi/PASCAL Code for the $\gamma_k[u]$-Algorithm. Appendix E shows a simple list of Appendix C for u = 6, …, 77.

For example, we generate the entries for u = 16. If we use (11) as initial term, we can get $\gamma_{11}[16], \ldots, \gamma_1[16]$ by using $\gamma_9[11], \ldots, \gamma_4[11]$ with (12), (13), (14) and m = 0.

$\gamma_{11}[16] = \gamma_9[11] = 961$
$\gamma_{10}[16] = \gamma_{11}[16] = 961$

$\gamma_9[16] = \gamma_{10}[16] - \gamma_8[11] = 961 - 194 = 767$
$\gamma_8[16] = \gamma_9[16] - \gamma_8[11] = 767 - 194 = 573$
$\gamma_7[16] = \gamma_8[16] - \gamma_8[11] = 573 - 194 = 379$

$\gamma_6[16] = 2\gamma_7[16] - \gamma_8[16] + \gamma_7[11] = 2\cdot379 - 573 + 43 = 228$
$\gamma_5[16] = 2\gamma_6[16] - \gamma_7[16] + \gamma_7[11] = 2\cdot228 - 379 + 43 = 120$
$\gamma_4[16] = 2\gamma_5[16] - \gamma_6[16] + \gamma_7[11] = 2\cdot120 - 228 + 43 = 55$

$\gamma_3[16] = 3\gamma_4[16] - 3\gamma_5[16] + \gamma_6[16] - \gamma_6[11] = 3\cdot55 - 3\cdot120 + 228 - 12 = 21$
$\gamma_2[16] = 3\gamma_3[16] - 3\gamma_4[16] + \gamma_5[16] - \gamma_5[11] = 3\cdot21 - 3\cdot 55 + 120 - 12 = 6$
$\gamma_1[16] = 3\gamma_2[16] - 3\gamma_3[16] + \gamma_4[16] - \gamma_4[11] = 3\cdot6 - 3\cdot21 + 55 - 9 = 1$

and with (8), (9), (10) we get the values of $\gamma_{12}[16]$, $\gamma_{13}[16]$ and $\gamma_{14}[16]$.

$$\gamma_{12}[16] = \sum_{k=1}^{11} \gamma_k[16] = 1 + 6 + 21 + 55 + 120 + 228 + 379 + 573 + 767 + 961 + 961 = 4072$$

$$\gamma_{13}[16] = \sum_{k=1}^{11} \gamma_k[16]\cdot(14-k) = 1\cdot13 + 6\cdot12 + 21\cdot11 + 55\cdot10 + 120\cdot9 + 228\cdot8 + 379\cdot7 + 573\cdot6 + 767\cdot5 + 9679\cdot3 + 967\cdot2 = 20423$$

$$\gamma_{14}[16] = \sum_{k=1}^{11} \frac{\gamma_k[16]\cdot(14-k)(19-k)}{2} = \frac{1}{2}\cdot 1\cdot13\cdot18 + 6\cdot12\cdot17 + 21\cdot11\cdot16 + 55\cdot10\cdot15 + 120\cdot9\cdot14 + 228\cdot8\cdot13 + 379\cdot7\cdot12 + 573\cdot6\cdot11 + 767\cdot5\cdot10 + 9679\cdot3\cdot9 + 967\cdot2\cdot8 = 108950$$

Now we have all entries for the row u = 16, which enable us to generate the entries for the row u = 21 and so forth.

u	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}	γ_{13}	γ_{14}
16	1	6	21	55	120	228	379	573	767	961	961	4072	20423	108950

12. How to get $B_n[u]$ for each u. Let $B^{m,k}[u]$ be the matrix in the *m*th row of each $S_m[u] \in G_{u-k}[u]$, defined by

$$\left.\begin{array}{c} B^{m,k}[u] := (b_{1,1} = s_{m,1} \;\cdots\; b_{1,u+1} = s_{m,u+1}), \quad S_m[u] \in G_{u-k}[u], \\ m = 3, \ldots, u-1 \quad \text{and} \quad k = 1, \ldots, u-5. \end{array}\right\} \qquad (15)$$

For example, for each u > 3, there exists u − 3 matrices $B^{m,1}[u]$.

When analyzing the sorted matrices $B_n[u]$ of each u > 3, the following relationships between the entries of $B^{m,1}[u-3]$ and $B^{3,k}[u]$ are identified. For each

$$\left.\begin{array}{c} B^{3,k}[u], \quad k = 2, \ldots, u-5, \\ \text{is} \quad (b_{1,1} \;\cdots\; b_{1,3}) = (1\ 1\ 0) \quad \text{and} \quad (b_{1,4} \;\cdots\; b_{1,u+1}) = B^{u-k-2,1}[u-3]. \end{array}\right\} \qquad (16)$$

Further is $$B^{3,1}[u] = B^{3,2}[u]. \qquad (17)$$

With other words, for k = 2, …, u − 5 the first three entries of each $B^{3,k}[u]$ are “1 1 0”, and the following u − 2 entries are the same as in $B^{u-k-2,1}[u-3]$.

For example, for u = 6 there are twelve matrices B_n. After sorting them we get three matrices S_m, sorted into one set G_t. For simplifying, we dispense with the parentheses of matrices. The shaded elements show A_m. The matrices S_m of the set G_t are listed from bottom to top.

	n	$G_5[6]$
$S_5[6]$	575	1 1 1 1 1 1 0
	287	1 1 1 1 1 0 1
	367	1 1 1 1 0 1 1
	999	1 1 1 0 1 1 1
	923	1 1 0 1 1 1 1
$S_4[6]$	735	1 1 1 1 1 0 0
	815	1 1 1 1 0 1 0
	423	1 1 1 0 1 1 0
	347	1 1 0 1 1 1 0
$S_3[6]$	975	1 1 1 1 0 0 1
	583	1 1 1 0 1 0 1
	507	1 1 0 1 1 0 1

For u = 6 we have three matrices $B^{m,1}[6]$, marked by the purple frame.

$$B^{5,1}[6] = B_{923}[6] = (1\ 1\ 0\ 1\ 1\ 1\ 1)$$
$$B^{4,1}[6] = B_{347}[6] = (1\ 1\ 0\ 1\ 1\ 1\ 0)$$
$$B^{3,1}[6] = B_{507}[6] = (1\ 1\ 0\ 1\ 1\ 0\ 1)$$

These three matrices build the elements after “1 1 0” of the matrices $B^{3,1}[9]$, $B^{3,2}[9]$, $B^{3,3}[9]$ and $B^{3,4}[9]$.

Compare it with the sorted matrices of u = 9 on page 12 to see how (16) and (17) work. Note the equal elements.

13. Same matrices $B_n[u]$ with *different* n. As seen in Appendix A for $\gamma_k[u] > 1$, there exist $\gamma_k[u]$ matrices $B_n[u]$ with *different* n, which is due to the reduction to the first u + 1 elements. Further, for each u > 6, the matrices of $G_{u-2}[u]$ exist $\gamma_2[u] + 1$ times, because $G_{u-2}[u] = \{G_{u-1}[u] \mid S_{u-1}[u] \notin G_{u-1}[u]\}$. In Chapter 15 we will more accurately respond to this special feature.

14. The $B_n[u]$-Algorithm. We conjecture, that exists an algorithm for each u ≥ 9, which generates each $B_n[u]$ by using $B^{m,1}[u-3]$, m = 3, …, u − 1.

First we build $S_3[u] \in G_{u-k}[u]$, k = 2, …, u − 5, by using (16) and (6) as shown by (18).

$$\left.\begin{aligned}
&S_3[u] \in G_{u-k}[u] = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & s_{1,7} & \cdots & s_{1,u+1} \\ 1 & 1 & 1 & 0 & 1 & 0 & s_{2,7} & \cdots & s_{2,u+1} \\ 1 & 1 & 0 & 1 & 1 & 0 & s_{3,7} & \cdots & s_{3,u+1} \end{pmatrix}, \quad k = 2, \dots, u-5 \\
&\text{with} \qquad (s_{3,4} \ \cdots \ s_{3,u+1}) = B^{u-k-2,1}[u-3] \\
&\text{and} \qquad (s_{3,6} \ \cdots \ s_{3,u+1}) = (s_{2,6} \ \cdots \ s_{2,u+1}) = (s_{1,6} \ \cdots \ s_{1,u+1}).
\end{aligned}\right\} \quad (18)$$

Then we build $S_{4,\,\ldots,\,u-k}[u] \in G_{u-k}[u]$, k = 2, …, u − 5, by using (7) as shown by (19).

$$\left.\begin{array}{l} S_m[u] \in G_{u-k}[u] = \begin{pmatrix} 1 & 1 & s_{1,3} & \cdots & s_{1,m+2} & 0 & s_{1,m+4} & \cdots & s_{1,u+1} \\ 1 & 1 & \vdots & \ddots & \vdots & 0 & \vdots & \ddots & \vdots \\ 1 & 1 & s_{m,3} & \cdots & s_{m,m+2} & 0 & s_{m,m+4} & \cdots & s_{m,u+1} \end{pmatrix}, \\ \begin{pmatrix} s_{1,3} & \cdots & s_{1,m+2} \\ \vdots & \ddots & \vdots \\ s_{m,3} & \cdots & s_{m,m+2} \end{pmatrix} = A_m, \quad m = 4, \ldots, u-k \quad \text{and} \quad k = 2, \ldots, u-5, \end{array}\right\} \quad (19)$$

until the biggest matrix $S_{u-k}[u] \in G_{u-k}[u]$ as shown by (20) is reached.

$$S_{u-k}[u] \in G_{u-k}[u] = \begin{pmatrix} 1 & 1 & s_{1,3} & \cdots & s_{1,m+2} & 0 & 0 & s_{1,m+5} & \cdots & s_{1,u+1} \\ 1 & 1 & \vdots & \ddots & \vdots & 0 & 0 & \vdots & \ddots & \vdots \\ 1 & 1 & s_{m,3} & \cdots & s_{m,m+2} & 0 & 0 & s_{m,m+5} & \cdots & s_{m,u+1} \end{pmatrix}. \quad (20)$$

The biggest set $G_{u-1}[u]$ is build as shown by (21).

$$G_{u-1}[u] = G_{u-2}[u], \quad S_{u-1}[u] \in G_{u-1}[u] = \begin{pmatrix} 1 & 1 & s_{1,3} & \cdots & s_{1,m+2} \\ 1 & 1 & \vdots & \ddots & \vdots \\ 1 & 1 & s_{m,3} & \cdots & s_{m,m+2} \end{pmatrix}. \quad (21)$$

At least, the number of sets $G_{u-k}[u]$ we get by (8) and the $\gamma_k[u]$-Algorithm.

15. Summary. With both algorithms we are in possession of tools to generate each $B_n[u]$ for $u \geq 11$. With the starting values of $\gamma_k[6]$, …, $\gamma_k[10]$ the $\gamma_k[u]$-Algorithm yields the numbers of matrices $B_n[u]$ and $S_m[u]$ and sets $G_t[u]$. The $B_n[u]$-Algorithm then yields the exact nature of matrices $B_n[u]$. (*reduced to the first u + 1 elements*)

But how to get each $n_z \in N_u$?

As mentioned in chapter 13, there exist same matrices $B_n[u]$ with *different* n_z for each $u \geq 6$. Remember a matrix $B_n[u]$ is a simplification for the distribution of the powers of 2 in $3^{u-i}2^{\alpha_i}$ in (3). Therefore, if (3) is interpreting as a Diophantine equation as shown by (22), then for each $B_n[u]$ the equation has $\gamma_k[u]$ different solutions $x = n_z$.

$$2y + 1 = \frac{3^u}{2^k} \cdot x + \sum_{i=1}^{u} \frac{3^{u-i} 2^{\alpha_i}}{2^k}, \quad k = \kappa(u) \quad (22)$$

Finally, the algorithms also allow an insight into the largely inscrutable properties of the number $\log_2 3$ = 1,5849625…, because they "replace" the expression $\kappa(u) = \lceil u \cdot \log_2 3 \rceil$ in (2) and (3) by elementary arithmetic. Most other work on the Collatz problem can not do without this mysterious number.

For example, the integer sequence by Finch [5] represents the numbers of $n_z \in N_u$ (values of $\gamma_{u-2}[u]$), and the integer sequence by Noe [2] represents N^*. Both sequences are generated by using $\log_2 3$ or the distinction log3 < log2. There is a Mathematica code for each sequence under the appropriate web-link.

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Appendix A. Complete list of sorted matrices $B_n[u]$ for u = 4, …, 9. For simplifying, we dispense with the parentheses of matrices. The matrices $S_m[u]$ of each set $G_t[u]$ are listed in columns from bottom to top. The shaded elements show A_m.

u = 4

	n	$G_3[4]$
$S_3[4]$	15	1 1 1 1 0
	7	1 1 1 0 1
	59	1 1 0 1 1

u = 5

	n	$G_4[5]$
$S_4[5]$	95	1 1 1 1 1 0
	175	1 1 1 1 0 1
	39	1 1 1 0 1 1
	219	1 1 0 1 1 1
$S_3[5]$	79	1 1 1 1 0 0
	199	1 1 1 0 1 0
	123	1 1 0 1 1 0

u = 6

	n	$G_5[6]$
$S_5[6]$	575	1 1 1 1 1 1 0
	287	1 1 1 1 1 0 1
	367	1 1 1 1 0 1 1
	999	1 1 1 0 1 1 1
	923	1 1 0 1 1 1 1
$S_4[6]$	735	1 1 1 1 1 0 0
	815	1 1 1 1 0 1 0
	423	1 1 1 0 1 1 0
	347	1 1 0 1 1 1 0
$S_3[6]$	975	1 1 1 1 0 0 1
	583	1 1 1 0 1 0 1
	507	1 1 0 1 1 0 1

---

u = 7

	n	$G_6[7]$
$S_6[7]$	383	1 1 1 1 1 1 1 0
	2239	1 1 1 1 1 1 0 1
	2975	1 1 1 1 1 0 1 1
	2031	1 1 1 1 0 1 1 1
	615	1 1 1 0 1 1 1 1
	2587	1 1 0 1 1 1 1 1
$S_5[7]$	1885	1 1 1 1 1 1 0 0
	1823	1 1 1 1 1 0 1 0
	1647	1 1 1 1 0 1 1 0
	3559	1 1 1 0 1 1 1 0
	2203	1 1 0 1 1 1 1 0
$S_4[7]$	4063	1 1 1 1 1 0 0 1
	3119	1 1 1 1 0 1 0 1
	1703	1 1 1 0 1 1 0 1
	3675	1 1 0 1 1 1 0 1
$S_3[7]$	1231	1 1 1 1 0 0 1 1
	3911	1 1 1 0 1 0 1 1
	1787	1 1 0 1 1 0 1 1
	59	1 1 0 1 1

n	$G_5[7]$	
1087	1 1 1 1 1 1 0 0	$S_5[7]$
2591	1 1 1 1 1 0 1 0	
879	1 1 1 1 0 1 1 0	
231	1 1 1 0 1 1 1 0	
1435	1 1 0 1 1 1 1 0	
3295	1 1 1 1 1 0 0 1	$S_4[7]$
2351	1 1 1 1 0 1 0 1	
935	1 1 1 0 1 1 0 1	
2907	1 1 0 1 1 1 0 1	
463	1 1 1 1 0 0 1 1	$S_3[7]$
3143	1 1 1 0 1 0 1 1	
1019	1 1 0 1 1 0 1 1	
59	1 1 0 1 1	

The matrix below the purple lines show $B^{3,1}[4]$. Note the equal elements.

For u = 8 the matrices $S_m[u]$ look something different. The green square frame show A_m. The matrix below a purple line show $B^{m,1}[u-3]$. Note the equal elements. The numbers n of equal sets $G_t[u]$ are not in special order.

u = 8

n, $G_7[8]$

	n										
$S_7[8]$	255	1	1	1	1	1	1	1	1	0
	4223	1	1	1	1	1	1	1	0	1
	1983	1	1	1	1	1	1	0	1	1
	6815	1	1	1	1	1	0	1	1	1
	5871	1	1	1	1	0	1	1	1	1
	4455	1	1	1	0	1	1	1	1	1
	2331	1	1	0	1	1	1	1	1	1
$S_6[8]$	3967	1	1	1	1	1	1	1	0	0
	1727	1	1	1	1	1	1	0	1	0
	6559	1	1	1	1	1	0	1	1	0
	5103	1	1	1	1	0	1	1	1	0
	4199	1	1	1	0	1	1	1	1	0
	539	1	1	0	1	1	1	1	1	0
$S_5[8]$	3903	1	1	1	1	1	1	0	0	1
	2079	1	1	1	1	1	0	1	0	1
	1135	1	1	1	1	0	1	1	0	1
	6375	1	1	1	0	1	1	1	0	1
	5787	1	1	0	1	1	1	1	0	1
$S_4[8]$	3039	1	1	1	1	1	0	0	1	1
	1071	1	1	1	1	0	1	0	1	1
	1191	1	1	1	0	1	1	0	1	1
	6747	1	1	0	1	1	1	0	1	1
$S_3[8]$	719	1	1	1	1	0	0	1	1	1
	6983	1	1	1	0	1	0	1	1	1
	4859	1	1	0	1	1	0	1	1	1
	219				1	1	0	1	1	1

n, $G_6[8]$

	n									
$S_6[8]$	3455	1	1	1	1	1	1	1	0	0
	1215	1	1	1	1	1	1	0	1	0
	6047	1	1	1	1	1	0	1	1	0
	5615	1	1	1	1	0	1	1	1	0
	3687	1	1	1	0	1	1	1	1	0
	1563	1	1	0	1	1	1	1	1	0
$S_5[8]$	5439	1	1	1	1	1	1	0	0	1
	1567	1	1	1	1	1	0	1	0	1
	7791	1	1	1	1	0	1	1	0	1
	7399	1	1	1	0	1	1	1	0	1
	4251	1	1	0	1	1	1	1	0	1
$S_4[8]$	3551	1	1	1	1	1	0	0	1	1
	2095	1	1	1	1	0	1	0	1	1
	7847	1	1	1	0	1	1	0	1	1
	5723	1	1	0	1	1	1	0	1	1
$S_3[8]$	207	1	1	1	1	0	0	1	1	1
	7495	1	1	1	0	1	0	1	1	1
	3835	1	1	0	1	1	0	1	1	1
	219				1	1	0	1	1	1

n, $G_6[8]$

	n									
$S_6[8]$	2431	1	1	1	1	1	1	1	0	0
	191	1	1	1	1	1	1	0	1	0
	5023	1	1	1	1	1	0	1	1	0
	4079	1	1	1	1	0	1	1	1	0
	2663	1	1	1	0	1	1	1	1	0
	2075	1	1	0	1	1	1	1	1	0
$S_5[8]$	4927	1	1	1	1	1	1	0	0	1
	543	1	1	1	1	1	0	1	0	1
	623	1	1	1	1	0	1	1	0	1
	7911	1	1	1	0	1	1	1	0	1
	5275	1	1	0	1	1	1	1	0	1
$S_4[8]$	2015	1	1	1	1	1	0	0	1	1
	2607	1	1	1	1	0	1	0	1	1
	679	1	1	1	0	1	1	0	1	1
	7259	1	1	0	1	1	1	0	1	1
$S_3[8]$	7375	1	1	1	1	0	0	1	1	1
	5959	1	1	1	0	1	0	1	1	1
	5371	1	1	0	1	1	0	1	1	1
	219				1	1	0	1	1	1

n, $G_5[8]$

	n									
$S_5[8]$	3135	1	1	1	1	1	1	0	0	0
	799	1	1	1	1	1	0	1	0	0
	7023	1	1	1	1	0	1	1	0	0
	5607	1	1	1	0	1	1	1	0	0
	4507	1	1	0	1	1	1	1	0	0
$S_4[8]$	2271	1	1	1	1	1	0	0	1	0
	303	1	1	1	1	0	1	0	1	0
	7079	1	1	1	0	1	1	0	1	0
	4955	1	1	0	1	1	1	0	1	0
$S_3[8]$	6607	1	1	1	1	0	0	1	1	0
	6215	1	1	1	0	1	0	1	1	0
	4091	1	1	0	1	1	0	1	1	0
	123				1	1	0	1	1	0

n, $G_5[8]$

	n									
$S_5[8]$	4159	1	1	1	1	1	1	0	0	0
	7967	1	1	1	1	1	0	1	0	0
	8047	1	1	1	1	0	1	1	0	0
	6631	1	1	1	0	1	1	1	0	0
	3483	1	1	0	1	1	1	1	0	0
$S_4[8]$	1247	1	1	1	1	1	0	0	1	0
	1327	1	1	1	1	0	1	0	1	0
	8103	1	1	1	0	1	1	0	1	0
	5979	1	1	0	1	1	1	0	1	0
$S_3[8]$	7631	1	1	1	1	0	0	1	1	0
	5191	1	1	1	0	1	0	1	1	0
	3067	1	1	0	1	1	0	1	1	0
	123				1	1	0	1	1	0

For u = 9 the matrices $S_m[u]$ look something different. The green square frame show A_m. The matrix below a purple line show $B^{m,1}[u - 3]$. Note the equal elements. The numbers n of equal sets $G_t[u]$ are not in special order.

u = 9

	n	$G_8[9]$									
$S_8[9]$	5631	1	1	1	1	1	1	1	1	1	0
	19199	1	1	1	1	1	1	1	1	0	1
	6783	1	1	1	1	1	1	1	0	1	1
	20927	1	1	1	1	1	1	0	1	1	1
	9375	1	1	1	1	1	0	1	1	1	1
	8431	1	1	1	1	0	1	1	1	1	1
	23399	1	1	1	0	1	1	1	1	1	1
	29467	1	1	0	1	1	1	1	1	1	1
$S_7[9]$	12543	1	1	1	1	1	1	1	1	0	0
	22655	1	1	1	1	1	1	1	0	1	0
	4031	1	1	1	1	1	1	0	1	1	0
	3743	1	1	1	1	1	0	1	1	1	0
	2799	1	1	1	1	0	1	1	1	1	0
	16743	1	1	1	0	1	1	1	1	1	0
	23835	1	1	0	1	1	1	1	1	1	0
$S_6[9]$	8063	1	1	1	1	1	1	1	0	0	1
	22207	1	1	1	1	1	1	0	1	0	1
	10655	1	1	1	1	1	0	1	1	0	1
	10735	1	1	1	1	0	1	1	1	0	1
	14439	1	1	1	0	1	1	1	1	0	1
	31771	1	1	0	1	1	1	1	1	0	1
$S_5[9]$	17727	1	1	1	1	1	1	0	0	1	1
	28703	1	1	1	1	1	0	1	0	1	1
	6255	1	1	1	1	0	1	1	0	1	1
	9959	1	1	1	0	1	1	1	0	1	1
	16027	1	1	0	1	1	1	1	0	1	1
$S_4[9]$	16863	1	1	1	1	1	0	0	1	1	1
	14895	1	1	1	1	0	1	0	1	1	1
	30887	1	1	1	0	1	1	0	1	1	1
	25691	1	1	0	1	1	1	0	1	1	1
$S_3[9]$	14031	1	1	1	1	0	0	1	1	1	1
	27975	1	1	1	0	1	0	1	1	1	1
	23803	1	1	0	1	1	0	1	1	1	1
	923				1	1	0	1	1	1	1

	n	$G_7[9]$									
$S_7[9]$	13567	1	1	1	1	1	1	1	1	0	0
	1151	1	1	1	1	1	1	1	0	1	0
	14271	1	1	1	1	1	1	0	1	1	0
	25247	1	1	1	1	1	0	1	1	1	0
	24303	1	1	1	1	0	1	1	1	1	0
	6503	1	1	1	0	1	1	1	1	1	0
	12571	1	1	0	1	1	1	1	1	1	0
$S_6[9]$	9087	1	1	1	1	1	1	1	0	0	1
	23231	1	1	1	1	1	1	0	1	0	1
	415	1	1	1	1	1	0	1	1	0	1
	9711	1	1	1	1	0	1	1	1	0	1
	24679	1	1	1	0	1	1	1	1	0	1
	20507	1	1	0	1	1	1	1	1	0	1
$S_5[9]$	7487	1	1	1	1	1	1	0	0	1	1
	7199	1	1	1	1	1	0	1	0	1	1
	27759	1	1	1	1	0	1	1	0	1	1
	20199	1	1	1	0	1	1	1	0	1	1
	26267	1	1	0	1	1	1	1	0	1	1
$S_4[9]$	15839	1	1	1	1	1	0	0	1	1	1
	15919	1	1	1	1	0	1	0	1	1	1
	29863	1	1	1	0	1	1	0	1	1	1
	3163	1	1	0	1	1	1	0	1	1	1
$S_3[9]$	13007	1	1	1	1	0	0	1	1	1	1
	28999	1	1	1	0	1	0	1	1	1	1
	1275	1	1	0	1	1	0	1	1	1	1
	923				1	1	0	1	1	1	1

n	$G_7[9]$										
2303	1	1	1	1	1	1	1	1	0	0	$S_7[9]$
127	1	1	1	1	1	1	1	0	1	0	
15295	1	1	1	1	1	1	0	1	1	0	
2719	1	1	1	1	1	0	1	1	1	0	
1775	1	1	1	1	0	1	1	1	1	0	
17767	1	1	1	0	1	1	1	1	1	0	
22811	1	1	0	1	1	1	1	1	1	0	
30591	1	1	1	1	1	1	1	0	0	1	$S_6[9]$
11967	1	1	1	1	1	1	0	1	0	1	
11679	1	1	1	1	1	0	1	1	0	1	
32239	1	1	1	1	0	1	1	1	0	1	
25703	1	1	1	0	1	1	1	1	0	1	
30747	1	1	0	1	1	1	1	1	0	1	
18751	1	1	1	1	1	1	0	0	1	1	$S_5[9]$
6175	1	1	1	1	1	0	1	0	1	1	
5231	1	1	1	1	0	1	1	0	1	1	
21223	1	1	1	0	1	1	1	0	1	1	
27291	1	1	0	1	1	1	1	0	1	1	
5599	1	1	1	1	1	0	0	1	1	1	$S_4[9]$
4655	1	1	1	1	0	1	0	1	1	1	
19623	1	1	1	0	1	1	0	1	1	1	
4187	1	1	0	1	1	1	0	1	1	1	
2767	1	1	1	1	0	0	1	1	1	1	$S_3[9]$
17735	1	1	1	0	1	0	1	1	1	1	
2299	1	1	0	1	1	0	1	1	1	1	
923				1	1	0	1	1	1	1	

	n	$G_6[9]$									
$S_6[9]$	22911	1	1	1	1	1	1	1	0	0	0
	27839	1	1	1	1	1	1	0	1	0	0
	26527	1	1	1	1	1	0	1	1	0	0
	24559	1	1	1	1	0	1	1	1	0	0
	30311	1	1	1	0	1	1	1	1	0	0
	12827	1	1	0	1	1	1	1	1	0	0
$S_5[9]$	23359	1	1	1	1	1	1	0	0	1	0
	21023	1	1	1	1	1	0	1	0	1	0
	20079	1	1	1	1	0	1	1	0	1	0
	3303	1	1	1	0	1	1	1	0	1	0
	9371	1	1	0	1	1	1	1	0	1	0
$S_4[9]$	21471	1	1	1	1	1	0	0	1	1	0
	20527	1	1	1	1	0	1	0	1	1	0
	11943	1	1	1	0	1	1	0	1	1	0
	18011	1	1	0	1	1	1	0	1	1	0
$S_3[9]$	27855	1	1	1	1	0	0	1	1	1	0
	10055	1	1	1	0	1	0	1	1	1	0
	17147	1	1	0	1	1	0	1	1	1	0
	347				1	1	0	1	1	1	0

	n	$G_6[9]$									
$S_6[9]$	23935	1	1	1	1	1	1	1	0	0	0
	5311	1	1	1	1	1	1	0	1	0	0
	16287	1	1	1	1	1	0	1	1	0	0
	15343	1	1	1	1	0	1	1	1	0	0
	6759	1	1	1	0	1	1	1	1	0	0
	3611	1	1	0	1	1	1	1	1	0	0
$S_5[9]$	831	1	1	1	1	1	1	0	0	1	0
	11807	1	1	1	1	1	0	1	0	1	0
	10863	1	1	1	1	0	1	1	0	1	0
	25831	1	1	1	0	1	1	1	0	1	0
	31899	1	1	0	1	1	1	1	0	1	0
$S_4[9]$	30687	1	1	1	1	1	0	0	1	1	0
	30767	1	1	1	1	0	1	0	1	1	0
	2727	1	1	1	0	1	1	0	1	1	0
	19035	1	1	0	1	1	1	0	1	1	0
$S_3[9]$	18639	1	1	1	1	0	0	1	1	1	0
	839	1	1	1	0	1	0	1	1	1	0
	6907	1	1	0	1	1	0	1	1	1	0
	347				1	1	0	1	1	1	0

n	$G_6[9]$										
13695	1	1	1	1	1	1	1	0	0	0	$S_6[9]$
4287	1	1	1	1	1	1	0	1	0	0	
25503	1	1	1	1	1	0	1	1	0	0	
25583	1	1	1	1	0	1	1	1	0	0	
7783	1	1	1	0	1	1	1	1	0	0	
13851	1	1	0	1	1	1	1	1	0	0	
32575	1	1	1	1	1	1	0	0	1	0	$S_5[9]$
22047	1	1	1	1	1	0	1	0	1	0	
21103	1	1	1	1	0	1	1	0	1	0	
2279	1	1	1	0	1	1	1	0	1	0	
8347	1	1	0	1	1	1	1	0	1	0	
31711	1	1	1	1	1	0	0	1	1	0	$S_4[9]$
29743	1	1	1	1	0	1	0	1	1	0	
12967	1	1	1	0	1	1	0	1	1	0	
8795	1	1	0	1	1	1	0	1	1	0	
28879	1	1	1	1	0	0	1	1	1	0	$S_3[9]$
11079	1	1	1	0	1	0	1	1	1	0	
16123	1	1	0	1	1	0	1	1	1	0	
347				1	1	0	1	1	1	0	

	n	$G_5[9]$									
$S_5[9]$	14399	1	1	1	1	1	1	0	0	0	1
	12063	1	1	1	1	1	0	1	0	0	1
	11119	1	1	1	1	0	1	1	0	0	1
	16871	1	1	1	0	1	1	1	0	0	1
	22939	1	1	0	1	1	1	1	0	0	1
$S_4[9]$	21727	1	1	1	1	1	0	0	1	0	1
	11567	1	1	1	1	0	1	0	1	0	1
	4007	1	1	1	0	1	1	0	1	0	1
	9051	1	1	0	1	1	1	0	1	0	1
$S_3[9]$	9679	1	1	1	1	0	0	1	1	0	1
	1095	1	1	1	0	1	0	1	1	0	1
	30715	1	1	0	1	1	0	1	1	0	1
	507				1	1	0	1	1	0	1

	n	$G_5[9]$									
$S_5[9]$	24639	1	1	1	1	1	1	0	0	0	1
	2847	1	1	1	1	1	0	1	0	0	1
	12143	1	1	1	1	0	1	1	0	0	1
	27111	1	1	1	0	1	1	1	0	0	1
	411	1	1	0	1	1	1	1	0	0	1
$S_4[9]$	12511	1	1	1	1	1	0	0	1	0	1
	21807	1	1	1	1	0	1	0	1	0	1
	26535	1	1	1	0	1	1	0	1	0	1
	10075	1	1	0	1	1	1	0	1	0	1
$S_3[9]$	19919	1	1	1	1	0	0	1	1	0	1
	24647	1	1	1	0	1	0	1	1	0	1
	7163	1	1	0	1	1	0	1	1	0	1
	507				1	1	0	1	1	0	1

n	$G_5[9]$										
23615	1	1	1	1	1	1	0	0	0	1	$S_5[9]$
13087	1	1	1	1	1	0	1	0	0	1	
1903	1	1	1	1	0	1	1	0	0	1	
26087	1	1	1	0	1	1	1	0	0	1	
32155	1	1	0	1	1	1	1	0	0	1	
22751	1	1	1	1	1	0	0	1	0	1	$S_4[9]$
20783	1	1	1	1	0	1	0	1	0	1	
2983	1	1	1	0	1	1	0	1	0	1	
32603	1	1	0	1	1	1	0	1	0	1	
18895	1	1	1	1	0	0	1	1	0	1	$S_3[9]$
2119	1	1	1	0	1	0	1	1	0	1	
8187	1	1	0	1	1	0	1	1	0	1	
507				1	1	0	1	1	0	1	

Appendix B. A Delphi/PASCAL Code for the $\gamma_k[u]$-Algorithm. Of course it is possible to program the algorithm even more compact. In any case, the PASCAL code is a simple way to understand how the algorithm works and also it is a good basis for translation into other languages.

```
const max = 100; // maximum limit of u

var
  u,k,i,b: integer;
  a,m: extended;
  g: array[-max..max, - max..max] of extended;

begin
  g[-max..max, - max..max]:=0 // set all arrays = 0 (Pseudocode)

  // u = 6..10 starting values
  g[1,6]:=1; g[2,6]:=1; g[3,6]:=3; g[4,6]:=12;
  g[1,7]:=1; g[2,7]:=1; g[3,7]:=2; g[4,7]:=7; g[5,7]:=30;
  g[1,8]:=1; g[2,8]:=2; g[3,8]:=2; g[4,8]:=5; g[5,8]:=19; g[6,8]:=85;
  g[1,9]:=1; g[2,9]:=2; g[3,9]:=3; g[4,9]:=3; g[5,9]:=9;  g[6,9]:=37; g[7,9]:=173;
  g[1,10]:=1; g[2,10]:=3; g[3,10]:=5; g[4,10]:=7; g[5,10]:=7; g[6,10]:=23; g[7,10]:=99;
g[8,10]:=476;

  u:=11;
  repeat
    m:=floor((u-6)/12);

    k:=5; a:=0; b:=1;
          repeat
            a:= a + g[u-12*b-5,u-12*b]*g[12*b-2,12*b];
            b:=b+1;
          until b>m;
          g[u-k,u]:= g[u-7,u-5] - a;
          g[u-k-1,u]:= g[u-5,u];

    k:=6; a:=0; b:=1;
          repeat
            a:= a + g[u-12*b-5,u-12*b]*g[12*b-3,12*b];
            b:=b+1;
          until b>m;
          repeat
            g[u-k-1,u]:= g[u-k,u] - g[u-8,u-5] + a;
            k:=k+1;
          until k>8;

    k:=8; a:=0; b:=1;
          repeat
            a:= a + g[u-12*b-5,u-12*b]*g[12*b-4,12*b];
            b:=b+1;
          until b>m;
          repeat
            g[u-k-2,u]:= -g[u-k,u] + 2*g[u-k-1,u] + g[u-9,u-5] - a;
            k:=k+1;
          until k>10;

    k:=10;
          repeat
            a:=0; b:=1;
            repeat
              a:= a + g[u-12*b-5,u-12*b]*g[12*b+5-k,12*b];
              b:=b+1;
            until b>m;
            g[u-k-3,u]:= g[u-k,u] - 3*g[u-k-1,u] + 3*g[u-k-2,u] - g[u-k,u-5] + a;
            k:=k+1;
          until k>12*(m+1)+1;

    i:=1; g[u-4,u]:=0; g[u-3,u]:=0; g[u-2,u]:=0;
    repeat
      g[u-4,u]:=g[u-4,u]+g[i,u];
      g[u-3,u]:=g[u-3,u]+g[i,u]*(u-i-2);
      g[u-2,u]:=g[u-2,u]+g[i,u]*(u-i-2)*(u-i+3)/2;
      i:=i+1;
    until i>u-5;

    u:=u+1;
  until u>max;

  // output of values as table (Pseudocode)
  u:=1;
  repeat
    list[ u, g[1,u], g[2,u], g[3,u], ... , g[max,u] ];
    u:=u+1;
  until u>max;
end;
```

Appendix C. Table of $\gamma_k[u]$ for u = 6, …, 30. Please use the zoom function for better reading.

u	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{15}	γ_{16}	γ_{17}	γ_{18}	γ_{19}	γ_{20}	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{25}	γ_{26}	γ_{27}	γ_{28}
6	1	1	3	12																								
7	1	1	2	7	30																							
8	1	2	2	5	19	85																						
9	1	2	3	3	9	37	173																					
10	1	3	5	7	7	23	99	476																				
11	1	3	6	9	12	12	43	194	961																			
12	1	4	9	16	23	30	30	113	525	2652																		
13	1	5	14	28	47	66	85	85	331	1570	8045																	
14	1	5	15	34	62	99	136	173	173	698	3387	17637																
15	1	6	20	50	103	179	278	377	476	476	1966	9690	51033															
16	1	6	21	55	120	228	379	573	767	961	961	4072	20423	108950														
17	1	7	27	77	180	366	665	1077	1602	2127	2652	2652	11433	58040	312455													
18	1	7	28	83	203	431	822	1431	2258	3303	4348	5393	5393	23701	121977	663535												
19	1	8	35	112	292	658	1323	2430	4122	6399	9261	12123	14985	14985	66734	346769	1900470											
20	1	9	44	154	434	1045	2233	4329	7749	12909	19809	28449	37089	45729	45729	205712	1076930	5936673										
21	1	9	45	164	483	1218	2721	5517	10304	17953	29335	44450	63298	82146	100994	100994	459632	2427209	13472296									
22	1	10	54	210	658	1757	4144	8833	17314	31553	53992	87073	130796	185161	239526	293891	293891	1348864	7168714	39993895								
23	1	10	55	219	705	1944	4746	10494	21338	40389	71719	120361	191348	284680	400357	516034	631711	631711	2927822	15676836	87986917							
24	1	11	65	275	933	2697	6890	15912	33765	66578	123132	214860	355847	560178	827853	1158872	1489891	1820910	1820910	8499580	45760565	257978502						
25	1	12	77	350	1265	3867	10389	25160	55844	115010	221702	403009	694065	1138049	1778140	2614338	3646643	4678948	5711253	5711253	26809375	144973012	820236724					
26	1	12	78	363	1350	4257	11805	29511	67679	144089	287384	540457	963838	1635694	2651829	4108047	6004348	8340732	10677116	13013500	13013500	61495590	334283132	1899474678				
27	1	13	90	442	1727	5702	16511	42987	102444	226435	468476	913736	1688727	2970994	4989115	8022701	12351363	17975101	24893915	31812729	38731543	38731543	183946295	1003880965	5723030586			
28	1	13	91	454	1807	6094	18061	48202	117855	267409	568583	1140777	2171495	3936768	6821577	11319853	18034477	27568330	39921412	55093723	70266034	85438345	85438345	408179706	2238139010	12809477536		
29	1	14	104	546	2273	7982	24563	67928	171842	402883	884183	1830602	3595335	6727952	12032438	20625233	33935232	53703785	81672242	117840603	162208868	206577133	250945398	250945398	1204202538	6626333161	38036848410	
30	1	14	105	559	2366	8460	26533	74855	193358	463287	1039766	2202672	4431210	8502189	15611999	27498588	46563439	75871570	119151534	180131884	258812620	355193742	451574864	547955986	547955986	2643267587	14606430265	84141805077

Appendix D. Relationships of $\gamma_k[u]$ for $u = 11, \ldots, 53$.

For $u = 11, \ldots, 17$

$$\gamma_{u-k}[u] = \gamma_{u-7}[u-5]\,, \quad k \in \{5\}$$
$$\gamma_{u-k}[u] - \gamma_{u-k-1}[u] = \gamma_{u-8}[u-5]\,, \quad k \in \{6, 7, 8\}$$
$$\gamma_{u-k}[u] - 2\gamma_{u-k-1}[u] + \gamma_{u-k-2}[u] = \gamma_{u-9}[u-5]\,, \quad k \in \{8, 9, 10\}$$
$$\gamma_{u-k}[u] - 3\gamma_{u-k-1}[u] + 3\gamma_{u-k-2}[u] - \gamma_{u-k-3}[u] = \gamma_{u-k}[u-5]\,, \quad k \in \{10, \ldots, u-4\}$$

For $u = 18, \ldots, 29$

$$\gamma_{u-k}[u] = \gamma_{u-7}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{10}[12]\,, \quad k \in \{5\}$$
$$\gamma_{u-k}[u] - \gamma_{u-k-1}[u] = \gamma_{u-8}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{9}[12]\,, \quad k \in \{6, 7, 8\}$$
$$\gamma_{u-k}[u] - 2\gamma_{u-k-1}[u] + \gamma_{u-k-2}[u] = \gamma_{u-9}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{8}[12]\,, \quad k \in \{8, 9, 10\}$$
$$\gamma_{u-k}[u] - 3\gamma_{u-k-1}[u] + 3\gamma_{u-k-2}[u] - \gamma_{u-k-3}[u] = \gamma_{u-k}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{17-k}[12]\,, \quad k \in \{10, \ldots, u-4\}$$

For $u = 30, \ldots, 41$

$$\gamma_{u-k}[u] = \gamma_{u-7}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{10}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{22}[24]\,, \quad k \in \{5\}$$
$$\gamma_{u-k}[u] - \gamma_{u-k-1}[u] = \gamma_{u-8}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{9}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{21}[24]\,, \quad k \in \{6, 7, 8\}$$
$$\gamma_{u-k}[u] - 2\gamma_{u-k-1}[u] + \gamma_{u-k-2}[u] = \gamma_{u-9}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{8}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{20}[24]\,, \quad k \in \{8, 9, 10\}$$
$$\gamma_{u-k}[u] - 3\gamma_{u-k-1}[u] + 3\gamma_{u-k-2}[u] - \gamma_{u-k-3}[u] = \gamma_{u-k}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{17-k}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{29-k}[24]\,, \quad k \in \{10, \ldots, u-4\}$$

For $u = 42, \ldots, 53$

$$\gamma_{u-k}[u] = \gamma_{u-7}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{10}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{22}[24] - \gamma_{u-41}[u-36] \cdot \gamma_{34}[36]\,, \quad k \in \{5\}$$
$$\gamma_{u-k}[u] - \gamma_{u-k-1}[u] = \gamma_{u-8}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{9}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{21}[24] - \gamma_{u-41}[u-36] \cdot \gamma_{33}[36]\,, \quad k \in \{6, 7, 8\}$$
$$\gamma_{u-k}[u] - 2\gamma_{u-k-1}[u] + \gamma_{u-k-2}[u] = \gamma_{u-9}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{8}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{20}[24] - \gamma_{u-41}[u-36] \cdot \gamma_{32}[36]\,, \quad k \in \{8, 9, 10\}$$
$$\gamma_{u-k}[u] - 3\gamma_{u-k-1}[u] + 3\gamma_{u-k-2}[u] - \gamma_{u-k-3}[u] = \gamma_{u-k}[u-5] - \gamma_{u-17}[u-12] \cdot \gamma_{17-k}[12] - \gamma_{u-29}[u-24] \cdot \gamma_{29-k}[24] - \gamma_{u-41}[u-36] \cdot \gamma_{41-k}[36]\,, \quad k \in \{10, \ldots, u-4\}$$

and so forth, but slightly modified.

Appendix E. Simple list of Appendix C for u = 6, …, 77.

6
1 1 3 12

7
1 1 2 7 30

8
1 2 2 5 19 85

9
1 2 3 3 9 37 173

10
1 3 5 7 7 23 99 476

11
1 3 6 9 12 12 43 194 961

12
1 4 9 16 23 30 30 113 525 2652

13
1 5 14 28 47 66 85 85 331 1570 8045

14
1 5 15 34 62 99 136 173 173 698 3387 17637

15
1 6 20 50 103 179 278 377 476 476 1966 9690 51033

16
1 6 21 55 120 228 379 573 767 961 961 4072 20423 108950

17
1 7 27 77 180 366 665 1077 1602 2127 2652 2652 11433 58040 312455

18
1 7 28 83 203 431 822 1431 2258 3303 4348 5393 5393 23701 121977 663535

19
1 8 35 112 292 658 1323 2430 4122 6399 9261 12123 14985 14985 66734 346769 1900470

20
1 9 44 154 434 1045 2233 4329 7749 12909 19809 28449 37089 45729 45729 205712 1076930 5936673

21
1 9 45 164 483 1218 2721 5517 10304 17953 29335 44450 63298 82146 100994 100994 459632 2427209 13472296

22
1 10 54 210 658 1757 4144 8833 17314 31553 53992 87073 130796 185161 239526 293891 293891 1348864 7168714 39993895

23
1 10 55 219 705 1944 4746 10494 21338 40389 71719 120361 191348 284680 400357 516034 631711 631711 2927822 15676836 87986917

24
1 11 65 275 933 2697 6890 15912 33765 66578 123132 214860 355847 560178 827853 1158872 1489891 1820910 1820910 8499580 45760565 257978502

25
1 12 77 350 1265 3867 10389 25160 55844 115010 221702 403009 694065 1138049 1778140 2614338 3646643 4678948 5711253 5711253 26809375 144973012 820236724

26
1 12 78 363 1350 4257 11805 29511 67679 144089 287384 540457 963838 1635694 2651829 4108047 6004348 8340732 10677116 13013500 13013500 61495590 334283132 1899474678

27
1 13 90 442 1727 5702 16511 42987 102444 226435 468476 913736 1688727 2970994 4989115 8022701 12351363 17975101 24893915 31812729 38731543 38731543 183946295 1003880965 5723030586

28
1 13 91 454 1807 6094 18061 48202 117855 267409 568583 1140777 2171495 3936768 6821577 11319853 18034477 27568330 39921412 55093723 70266034 85438345 85438345 408179706 2238139010 12809477536

29
1 14 104 546 2273 7982 24563 67928 171842 402883 884183 1830602 3595335 6727952 12032438 20625233 33935232 53703785 81672242 117840603 162208868 206577133 250945398 250945398 1204202538 6626333161 38036848410

30
1 14 105 559 2366 8460 26533 74855 193358 463287 1039766 2202672 4431210 8502189 15611999 27498588 46563439 75871570 119151534 180131884 258812620 355193742 451574864 547955986 547955986 2643267587 14606430265 84141805077

31
1 15 119 665 2938 10920 35483 103441 275583 680086 1570569 3420911 7071818 13944124 26318826 47683853 83080835 139451872 225639534 352386861 530436893 759789630 1040445072 1321100514 1601755956 1601755956 7756962475 43000123284 248369601964

32
1 16 135 798 3710 14443 48958 148376 409747 1045726 2492965 5597670 11915412 24170921 46923592 87438703 156764345 270808352 451415231 726366162 1129378998 1694171592 2420743944 3309096054 4197448164 5085800274 5085800274 24708004563 137322066880 794919136728

33
1 16 136 815 3857 15323 53079 164502 464648 1212835 2956976 6790112 14782443 30671003 60907973 116168626 213318904 377842627 646268702 1068598332 1708305016 2642334549 3947632726 5624199547 7672035012 9719870477 11767705942 11767705942 57390010121 319962207200 1857112329035

34
1 17 152 952 4690 19341 69391 222313 648024 1742876 4372173 10318287 23065169 49104416 100002420 195522490 368095838 668641429 1173734695 1992776249 3275160599 5210276148 8027505194 11956230035 16996450671 23148167102 29299883533 35451599964 35451599964 173405214133 969115652680 5636545892795

35
1 17 153 968 4828 20193 73584 239724 711191 1947972 4978637 11973494 27281073 59212619 122974951 245267721 471176784 873995390 1567973198 2723993112 4585248117 7482918115 11836169925 18152157283 26938033925 38193799851 51919455061 65645110271 79370765481 79370765481 389606249120 2183762223357 12732900345928

36
1 18 170 1122 5812 25160 94649 317690 969536 2728323 7156370 17646598 41189915 91523745 194489554 396758245 779413313 1478212670 2712350050 4821715994 8312658415 13903743163 22571514590 35550495550 54333187399 80412091493 113787207832 154458536416 195129865000 235801193584 235801193584 1160606285961 6519793249890 38088111350198

37
1 19 189 1309 7089 31972 124916 434297 1369777 3976147 10740847 27238172 65304229 148880694 324305509 677664857 1362853058 2645019334 4965112696 9031236206 15936810614 27305547370 45436421636 73448645298 115281666966 175695171974 259448845656 366542688012 496976699042 627410710072 757844721102 757844721102 3738436950162 21039916860397 123110229387834

38
1 19 190 1329 7293 33405 132744 469812 1509257 4463741 12288337 31761912 77619334 180370197 400455893 852853078 1748127880 3458281581 6618064889 12274774300
22099544050 38660133657 65755211053 108748635716 174903741802 273383338145 415249708257 611565135650 862329620324 1167543162279 1472756704234 1777970246189
1777970246189 8794596642737 49608052563411 290838337577435

39
1 20 209 1520 8643 40919 167739 611416 2019974 6136289 17332058 45920332 114935340 273350598 620724418 1351295518 2829880096 5717545469 11171954192 21154191386
38880806818 69459803276 120710571239 204167769842 336085207841 538439724578 838931189981 1270982504564 1868016568841 2630033382812 3557032946477 4484032510142
5411032073807 5411032073807 26822143250162 151564955142510 889949312454085

40
1 20 210 1539 8835 42291 175491 648159 2171486 6693444 19192670 51641697 131311027 317350814 732471734 1621044895 3451655264 7091676941 14093791692 27149544464
50780201916 92350352311 163489685114 281922950296 473708097344 775474414271 1236660666626 1919515097494 2899095427496 4250459377253 5973606946765 8068538136032
10163469325299 12258400514566 12258400514566 60920541130023 344986952773120 2029460152095008

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